

HOLONOMIC COMPLEXES ON ABELIAN VARIETIES, PART I

CHRISTIAN SCHNELL

ABSTRACT. We study the Fourier-Mukai transform for holonomic \mathcal{D} -modules on a complex abelian variety. Among other things, we show that the cohomology support loci of a holonomic complex are finite unions of translates of triple tori, the translates being by torsion points for objects of geometric origin; and that the standard t-structure on the derived category of holonomic complexes corresponds, under the Fourier-Mukai transform, to a certain perverse coherent t-structure in the sense of Kashiwara and Arinkin-Bezrukavnikov.

A. OVERVIEW OF THE PAPER

1. Introduction. This is the first in a series of papers about holonomic \mathcal{D} -modules on complex abelian varieties; the ultimate goal of the series is, in a nutshell, the description of holonomic \mathcal{D} -modules in terms of the Fourier-Mukai transform.

Let A be a complex abelian variety, and let \mathcal{D}_A be the sheaf of linear differential operators. The most basic examples of left \mathcal{D}_A -modules are line bundles L with integrable connection $\nabla: L \rightarrow \Omega_A^1 \otimes L$. Because A is an abelian variety, the moduli space A^\natural of all such is a quasi-projective algebraic variety of dimension $2 \dim A$. The main idea in the study of \mathcal{D}_A -modules is to exploit the fact that A^\natural is so big.

One approach is to consider, for a left \mathcal{D}_A -module \mathcal{M} , the cohomology groups (in the sense of \mathcal{D} -modules) of the various twists $\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla)$. This information may be presented using the *cohomology support loci* of \mathcal{M} , which are the sets

$$(1.1) \quad S_m^k(\mathcal{M}) = \left\{ (L, \nabla) \in A^\natural \mid \dim \mathbf{H}^k \left(A, \mathrm{DR}_A(\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla)) \right) \geq m \right\}.$$

Another way to present the information about cohomology of twists of \mathcal{M} is through the *Fourier-Mukai transform* for algebraic \mathcal{D}_A -modules, which was introduced and studied by Laumon [Lau96] and Rothstein [Rot96]. It is an exact functor

$$(1.2) \quad \mathrm{FM}_A: \mathrm{D}_{coh}^b(\mathcal{D}_A) \rightarrow \mathrm{D}_{coh}^b(\mathcal{O}_{A^\natural}),$$

defined as the integral transform with kernel $(P^\natural, \nabla^\natural)$, the tautological line bundle with relative integrable connection on $A \times A^\natural$. As shown by Laumon and Rothstein, FM_A is an equivalence between the bounded derived category of coherent algebraic \mathcal{D}_A -modules, and that of algebraic coherent sheaves on A^\natural . In essence, this means that an algebraic \mathcal{D} -module on an abelian variety can be recovered from the cohomology of its twists by line bundles with integrable connection.

Now the most interesting \mathcal{D} -modules are clearly the holonomic ones; recall that a \mathcal{D} -module is *holonomic* if its characteristic variety is a Lagrangian subset of the cotangent bundle. One reason is that, via the Riemann-Hilbert correspondence, the category of regular holonomic \mathcal{D} -modules is equivalent to the category of perverse sheaves. The motivation for this study is the following natural question:

Question. Let $\mathrm{D}_h^b(\mathcal{D}_A)$ denote the full subcategory of $\mathrm{D}_{coh}^b(\mathcal{D}_A)$, consisting of complexes with holonomic cohomology sheaves. What is the image of $\mathrm{D}_h^b(\mathcal{D}_A)$ under

Key words and phrases. Abelian variety, D-module, holonomic complex, constructible complex, Fourier-Mukai transform, cohomology support loci, perverse coherent sheaf.

the Fourier-Mukai transform? In particular, what does the complex of sheaves $\mathrm{FM}_A(\mathcal{M})$ look like when \mathcal{M} is a holonomic \mathcal{D}_A -module?

In this paper, we prove several results about cohomology support loci and Fourier-Mukai transforms of holonomic complexes of \mathcal{D}_A -modules. Among other things, we establish a fundamental structure theorem for cohomology support loci, and show that the Fourier-Mukai transform of a holonomic \mathcal{D}_A -module satisfies certain codimension inequalities that very much resemble the famous generic vanishing theorem of Green and Lazarsfeld [GL87].

In fact, the similarities with generic vanishing theory are no accident. As explained in [PS11], generic vanishing theory is really a collection of statements about certain holonomic \mathcal{D} -modules on abelian varieties (namely those that are obtained as direct images of structure sheaves of irregular smooth projective varieties) and natural filtrations on them; quite surprisingly, it turns out that all statements that do not involve the filtration are actually true for arbitrary holonomic \mathcal{D} -modules. This study should therefore be viewed as a continuation and generalization of the work of Green-Lazarsfeld [GL87, GL91], Arapura [Ara92], and Simpson [Sim93].

Note. Before we proceed, a few remarks about related works may be helpful.

- (1) Some of the results in this paper have been announced in [PS11, §26].
- (2) The main object of [PS11] are filtered \mathcal{D}_A -modules underlying mixed Hodge modules on A , and in particular, the individual coherent sheaves in the filtration. We do not prove any result of that type in this paper.
- (3) For regular holonomic \mathcal{D} -modules (or equivalently, for perverse sheaves), similar but generally weaker statements were also obtained by Krämer and Weissauer [KW11, Wei12].
- (4) All our results here also hold, suitably interpreted, for holonomic \mathcal{D} -modules on compact complex tori. This will be explained elsewhere.

In Part II of the series, we will show (using the work of Mochizuki and Sabbah) that the Fourier-Mukai transform of a semisimple holonomic \mathcal{D}_A -module is locally analytically represented by a complex with linear differentials. In Part III, we shall (hopefully) answer the above question, by identifying the image of the category of holonomic \mathcal{D}_A -modules with a suitably defined category of “hyperkähler perverse sheaves” on the noncompact hyperkähler manifold A^\natural .

2. Results about constructible complexes. Although the focus of this work is on holonomic \mathcal{D} -modules on abelian varieties, we shall begin by describing the main results in the more familiar setting of constructible complexes. Proofs for all the theorems in this section may be found in §15.

First a few words about the terminology. By a *constructible complex* on the abelian variety A , we mean a complex E of sheaves of \mathbb{C} -vector spaces, whose cohomology sheaves $\mathcal{H}^i E$ are constructible with respect to an algebraic (or equivalently, complex analytic) stratification of A , and vanish for i outside some bounded interval. We denote by $D_c^b(\mathbb{C}_A)$ the bounded derived category of constructible complexes. A basic fact [HTT08, Section 4.5] is that the hypercohomology groups $\mathbf{H}^i(A, E)$ are finite-dimensional complex vector spaces for any $E \in D_c^b(\mathbb{C}_A)$.

Now let $\mathrm{Char}(A)$ be the space of rank one characters of the fundamental group; it is also the moduli space for local systems of rank one, and for any character $\rho: \pi_1(A) \rightarrow \mathbb{C}^*$, we denote the corresponding local system on A by the symbol \mathbb{C}_ρ . It is easy to see that $E \otimes_{\mathbb{C}} \mathbb{C}_\rho$ is again constructible for any $E \in D_c^b(\mathbb{C}_A)$; we may therefore define the *cohomology support loci* of $E \in D_c^b(\mathbb{C}_A)$ to be the subsets

$$(2.1) \quad S_m^k(E) = \left\{ \rho \in \mathrm{Char}(A) \mid \dim \mathbf{H}^k(A, E \otimes_{\mathbb{C}} \mathbb{C}_\rho) \geq m \right\},$$

for any pair of integers $k, m \in \mathbb{Z}$. Since the space of rank one characters is very large – its dimension is equal to $2 \dim A$ – these loci should contain a lot of information about the original constructible complex E , and indeed they do.

Our first result is the following structure theorem for cohomology support loci.

Theorem 2.2. *Let $E \in D_c^b(\mathbb{C}_A)$ be a constructible complex.*

- (a) *Each $S_m^k(E)$ is a finite union of linear subvarieties of $\text{Char}(A)$.*
- (b) *If E is a semisimple perverse sheaf of geometric origin [BBD82, 6.2.4], then these linear subvarieties are arithmetic.*

Here we are using the expression (*arithmetic*) *linear subvarieties* for what Simpson was originally calling (*torsion*) *translates of triple tori* in [Sim93, p. 365]; the precise definition is the following.

Definition 2.3. A *linear subvariety* of $\text{Char}(A)$ is any subset of the form

$$(2.4) \quad \rho \cdot \text{im}(\text{Char}(f): \text{Char}(B) \rightarrow \text{Char}(A)),$$

for a surjective morphism of abelian varieties $f: A \rightarrow B$ with connected fibers, and a character $\rho \in \text{Char}(A)$. We say that a linear subvariety is *arithmetic* if ρ can be taken to be torsion point of $\text{Char}(A)$.

Note. The reason for the term *arithmetic* is as follows. Let A^\natural be the moduli space of line bundles with integrable connection on A ; it is also a complex algebraic variety, biholomorphic to $\text{Char}(A)$, but with a different algebraic structure. When A is defined over a number field, the torsion points are precisely those points on the algebraic varieties $\text{Char}(A)$ and A^\natural that are simultaneously defined over a number field in both [Sim93, Proposition 3.4].

Our next result has to do with the codimension of the cohomology support loci $S^k(E) = S_1^k(E)$. Recall that the category $D_c^b(\mathbb{C}_A)$ has a nonstandard t-structure

$$\left({}^\pi D_c^{\leq 0}(\mathbb{C}_A), {}^\pi D_c^{\geq 0}(\mathbb{C}_A) \right),$$

called the *perverse t-structure*, whose heart is the abelian category of perverse sheaves [BBD82]. We show that the position of a constructible complex with respect to this t-structure can be read off from its cohomology support loci.

Theorem 2.5. *Let $E \in D_c^b(\mathbb{C}_A)$ be a constructible complex.*

- (a) *One has $E \in {}^\pi D_c^{\leq 0}(\mathbb{C}_A)$ iff $\text{codim } S^k(E) \geq 2k$ for every $k \in \mathbb{Z}$.*
- (b) *Similarly, $E \in {}^\pi D_c^{\geq 0}(\mathbb{C}_A)$ iff $\text{codim } S^k(E) \geq -2k$ for every $k \in \mathbb{Z}$.*
- (c) *In particular, E is a perverse sheaf iff $\text{codim } S^k(E) \geq |2k|$ for every $k \in \mathbb{Z}$.*

A consequence is the following “generic vanishing theorem” for perverse sheaves; a similar (but less precise) statement has also been proved recently by Krämer and Weissauer [KW11, Theorem 2].

Corollary 2.6. *Let $E \in D_c^b(\mathbb{C}_A)$ be a perverse sheaf on a complex abelian variety. Then the cohomology support loci $S_m^k(E)$ are finite unions of linear subvarieties of $\text{Char}(A)$ of codimension at least $|2k|$. In particular, one has*

$$\mathbf{H}^k(A, E \otimes_{\mathbb{C}} \mathbb{C}_\rho) = 0$$

for general $\rho \in \text{Char}(A)$ and $k \neq 0$.

The generic vanishing theorem implies that the *Euler characteristic*

$$\chi(A, E) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathbf{H}^k(A, E)$$

of a perverse sheaf on an abelian variety is always nonnegative, a result originally due to Franek and Kapranov [FK00, Corollary 1.4]. Indeed, from the deformation invariance of the Euler characteristic, we get

$$\chi(A, E) = \chi(A, E \otimes_{\mathbb{C}} \mathbb{C}_{\rho}) = \dim \mathbf{H}^0(A, E \otimes_{\mathbb{C}} \mathbb{C}_{\rho}) \geq 0$$

for a generic character $\rho \in \text{Char}(A)$. For *simple* perverse sheaves with $\chi(A, E) = 0$, we have the following structure theorem, which has also been proved by Weissauer [Wei12, Theorem 2].

Theorem 2.7. *Let $E \in \mathbf{D}_c^b(\mathbb{C}_A)$ be a simple perverse sheaf. If $\chi(A, E) = 0$, then there exists an abelian variety B , a surjective morphism $f: A \rightarrow B$ with connected fibers, and a simple perverse sheaf $E' \in \mathbf{D}_c^b(\mathbb{C}_B)$ with $\chi(B, E') > 0$, such that*

$$E = f^* E' \otimes_{\mathbb{C}} \mathbb{C}_{\rho}$$

for some character $\rho \in \text{Char}(A)$.

3. Results about holonomic complexes. All the theorems in the previous section are actually consequences of similar results about holonomic complexes of \mathcal{D} -modules on abelian varieties. In fact, the situation for \mathcal{D}_A -modules is considerably better, because we have the Fourier-Mukai transform (1.2) at our disposal.

Again, we begin by saying a few words about terminology. Recall that \mathcal{D}_A is the sheaf of linear differential operators of finite order; since the tangent bundle of A is trivial, \mathcal{D}_A is generated, as an \mathcal{O}_A -algebra, by any basis $\partial_1, \dots, \partial_g \in H^0(A, \mathcal{T}_A)$, subject to the relations $[\partial_i, \partial_j] = 0$ and $[\partial_i, f] = \partial_i f$. By an (algebraic) \mathcal{D}_A -module, we mean a sheaf of left \mathcal{D}_A -modules that is quasi-coherent as a sheaf of \mathcal{O}_A -modules; a \mathcal{D}_A -module is holonomic if its characteristic variety, as a subset of the cotangent bundle T^*A , has dimension equal to $\dim A$. Finally, a *holonomic complex* is a complex of \mathcal{D}_A -modules \mathcal{M} , whose cohomology sheaves $\mathcal{H}^i \mathcal{M}$ are holonomic, and vanish for i outside some bounded interval. We denote by $\mathbf{D}_{coh}^b(\mathcal{D}_A)$ the bounded derived category of coherent \mathcal{D}_A -modules, and by $\mathbf{D}_h^b(\mathcal{D}_A)$ the full subcategory of holonomic complexes.

Let A^{\natural} be the moduli space of line bundles with integrable connection on A . For any \mathcal{D}_A -module \mathcal{M} , and any $(L, \nabla) \in A^{\natural}$, the tensor product $\mathcal{M} \otimes_{\mathcal{O}_A} L$ again has the structure of a \mathcal{D}_A -module; for the sake of clarity, we will denote it by the symbol $\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla)$. We then define the cohomology support loci of a complex of \mathcal{D}_A -modules $\mathcal{M} \in \mathbf{D}_{coh}^b(\mathcal{D}_A)$ by the same formula as in (1.1), where DR_A denotes the de Rham functor.

One of the main results of this paper is the following structure theorem for cohomology support loci of holonomic complexes.

Definition 3.1. A *linear subvariety* of A^{\natural} is any subset of the form

$$(3.2) \quad (L, \nabla) \otimes \text{im}(f^{\natural}: B^{\natural} \rightarrow A^{\natural}),$$

for a surjective morphism of abelian varieties $f: A \rightarrow B$ with connected fibers, and a line bundle with integrable connection $(L, \nabla) \in A^{\natural}$. We say that a linear subvariety is *arithmetic* if (L, ∇) can be taken to be torsion point of A^{\natural} .

Theorem 3.3. *Let $\mathcal{M} \in \mathbf{D}_h^b(\mathcal{D}_A)$ be a holonomic complex.*

- (a) *Each $S_m^k(\mathcal{M})$ is a finite union of linear subvarieties of A^{\natural} .*
- (b) *If \mathcal{M} is a semisimple regular holonomic \mathcal{D}_A -module of geometric origin, in the sense of [BBD82, 6.2.4], then these linear subvarieties are arithmetic.*

Theorem 3.3 and Theorem 2.2 are proved together; in fact, the main idea is to exploit the close relationship between cohomology support loci for constructible and holonomic complexes. Recall that we have a biholomorphic mapping

$$(3.4) \quad \Phi: A^{\natural} \rightarrow \text{Char}(A), \quad (L, \nabla) \mapsto \ker \nabla,$$

by the well-known correspondence between local systems and vector bundles with integrable connection. Now if \mathcal{M} is a holonomic \mathcal{D}_A -module, then according to a theorem of Kashiwara [HTT08, Theorem 4.6.6], its de Rham complex

$$\mathrm{DR}_A(\mathcal{M}) = [\mathcal{M} \rightarrow \Omega_A^1 \otimes \mathcal{M} \rightarrow \cdots \rightarrow \Omega_A^{\dim A} \otimes \mathcal{M}],$$

placed in degrees $-\dim A, \dots, 0$, is a perverse sheaf on A . More generally, $\mathrm{DR}_A(\mathcal{M})$ is a constructible complex for any $\mathcal{M} \in \mathrm{D}_h^b(\mathcal{D}_A)$ [HTT08, Theorem 4.6.3]. It is very easy to show – see Lemma 11.1 below – that the cohomology support loci for \mathcal{M} and $\mathrm{DR}_A(\mathcal{M})$ are connected by the formula

$$(3.5) \quad \Phi(S_m^k(\mathcal{M})) = S_m^k(\mathrm{DR}_A(\mathcal{M})).$$

A much deeper relationship comes from the Riemann-Hilbert correspondence of Kashiwara and Mebkhout [HTT08, Theorem 7.2.1], which asserts that the functor

$$\mathrm{DR}_A: \mathrm{D}_{rh}^b(\mathcal{D}_A) \rightarrow \mathrm{D}_c^b(\mathbb{C}_A)$$

from regular holonomic complexes to constructible complexes is an equivalence of categories. Together with (3.5), this means that cohomology support loci for holonomic and constructible complexes completely determine each other.

Let us now briefly sketch the proof of Theorem 3.3. Our starting point is the observation that both complex manifolds $\mathrm{Char}(A)$ and A^\natural naturally have the structure of complex algebraic varieties; while isomorphic as complex manifolds, they are not isomorphic as algebraic varieties. The special property of *linear subvarieties* is that they are algebraic in both models. Indeed, for any surjective morphism $f: A \rightarrow B$ of abelian varieties, (3.2) is an algebraic subvariety of A^\natural , and (2.4) is an algebraic subvariety of $\mathrm{Char}(A)$; moreover, since Φ is an isomorphism of complex Lie groups,

$$\Phi\left((L, \nabla) \otimes \mathrm{im}(f^\natural: B^\natural \rightarrow A^\natural)\right) = \Phi(L, \nabla) \cdot \mathrm{im}(\mathrm{Char}(f): \mathrm{Char}(B) \rightarrow \mathrm{Char}(A)).$$

The following difficult theorem by Simpson [Sim93, Theorem 3.1] shows that finite unions of linear subvarieties are the only closed subsets with this property.

Theorem 3.6 (Simpson). *Let Z be a closed algebraic subset of A^\natural . If $\Phi(Z)$ is again a closed algebraic subset of $\mathrm{Char}(A)$, then Z is a finite union of linear subvarieties of A^\natural , and $\Phi(Z)$ is a finite union of linear subvarieties of $\mathrm{Char}(A)$.*

Thus it suffices to show that cohomology support loci are algebraic subsets of $\mathrm{Char}(A)$ and A^\natural , which we do in Theorem 10.6 and Proposition 11.3 below. The argument in §10 is based on a sort of “Fourier-Mukai transform” for constructible complexes, which may be of independent interest.

Theorem 3.3 has the following consequence for the support of the Fourier-Mukai transform of a holonomic complex.

Corollary 3.7. *Let $\mathcal{M} \in \mathrm{D}_h^b(\mathcal{D}_A)$ be a holonomic complex on an abelian variety. Then the support of the complex $\mathrm{FM}_A(\mathcal{M})$ is a finite union of linear subvarieties of A^\natural . These linear subvarieties are arithmetic whenever \mathcal{M} is a semisimple regular holonomic \mathcal{D}_A -module of geometric origin.*

Our second main result is that the position of a holonomic complex \mathcal{M} with respect to the standard t-structure on $\mathrm{D}_h^b(\mathcal{D}_A)$ can be read off from the codimension of its cohomology support loci $S^k(\mathcal{M}) = S_1^k(\mathcal{M})$.

Theorem 3.8. *Let $\mathcal{M} \in \mathrm{D}_h^b(\mathcal{D}_A)$ be a holonomic complex.*

- (a) *One has $\mathcal{M} \in \mathrm{D}_h^{\leq 0}(\mathcal{D}_A)$ iff $\mathrm{codim} S^k(\mathcal{M}) \geq 2k$ for every $k \in \mathbb{Z}$.*
- (b) *Similarly, $\mathcal{M} \in \mathrm{D}_h^{\geq 0}(\mathcal{D}_A)$ iff $\mathrm{codim} S^k(\mathcal{M}) \geq -2k$ for every $k \in \mathbb{Z}$.*
- (c) *In particular, \mathcal{M} is a single holonomic \mathcal{D}_A -module iff $\mathrm{codim} S^k(\mathcal{M}) \geq |2k|$ for every $k \in \mathbb{Z}$.*

The proof (in §14) uses the structure theorem above, together with certain properties of the Fourier-Mukai transform established by Laumon. Theorem 3.8 can be reformulated using the theory of *perverse coherent sheaves*, developed by Kashiwara [Kas04] and by Arinkin and Bezrukavnikov [AB10]. In fact, there is a perverse t-structure on $D_{coh}^b(\mathcal{O}_{A^\natural})$ with the property that

$${}^m D_{coh}^{\leq 0}(\mathcal{O}_{A^\natural}) = \{ E \in D_{coh}^b(\mathcal{O}_{A^\natural}) \mid \text{codim Supp } \mathcal{H}^k E \geq 2k \text{ for every } k \in \mathbb{Z} \};$$

it corresponds to the supporting function $m = \lfloor \frac{1}{2} \text{codim} \rfloor$ on the topological space of the scheme A^\natural , in Kashiwara's terminology. Its heart ${}^m \text{Coh}(\mathcal{O}_{A^\natural})$ is the abelian category of *m-perverse coherent sheaves*.

Theorem 3.9. *Let $\mathcal{M} \in D_h^b(\mathcal{D}_A)$ be a holonomic complex on A .*

- (a) *One has $\mathcal{M} \in D_h^{\leq k}(\mathcal{D}_A)$ iff $\text{FM}_A(\mathcal{M}) \in {}^m D_{coh}^{\leq k}(\mathcal{O}_{A^\natural})$.*
- (b) *Similarly, $\mathcal{M} \in D_h^{\geq k}(\mathcal{D}_A)$ iff $\text{FM}_A(\mathcal{M}) \in {}^m D_{coh}^{\geq k}(\mathcal{O}_{A^\natural})$.*
- (c) *In particular, \mathcal{M} is a single holonomic \mathcal{D}_A -module iff its Fourier-Mukai transform $\text{FM}_A(\mathcal{M})$ is an m-perverse coherent sheaf on A^\natural .*

Using basic properties of the *m*-perverse t-structure (see §13), it follows that the Fourier-Mukai transform of a holonomic \mathcal{D}_A -module is always concentrated in degrees $0, 1, \dots, \dim A$. Moreover, $\mathcal{H}^i \text{FM}_A(\mathcal{M})$ is a torsion sheaf for $i > 0$; and for $i = 0$, it is a torsion sheaf iff it is zero.

For that reason, one would expect $\mathcal{H}^0 \text{FM}_A(\mathcal{M})$ to be supported on all of A^\natural , but examples show that this fails when \mathcal{M} is pulled back from a lower-dimensional abelian variety. This suggests our third main result, namely the following structure theorem for *simple* holonomic \mathcal{D}_A -modules.

Theorem 3.10. *Let \mathcal{M} be a simple holonomic \mathcal{D}_A -module. Then there exists an abelian variety B , a surjective morphism $f: A \rightarrow B$ with connected fibers, and a simple holonomic \mathcal{D}_B -module \mathcal{N} with $\text{Supp } \mathcal{H}^0 \text{FM}_B(\mathcal{N}) = B^\natural$, such that*

$$\mathcal{M} \simeq f^* \mathcal{N} \otimes_{\mathcal{O}_A} (L, \nabla)$$

for some $(L, \nabla) \in A^\natural$.

This result again follows from the basic properties of the Fourier-Mukai transform, together with the following interesting fact about the *m*-perverse t-structure (see Proposition 13.6): If a complex of coherent sheaves E and its dual complex $\mathbf{R}\text{Hom}(E, \mathcal{O}_{A^\natural})$ both belong to ${}^m \text{Coh}(\mathcal{O}_{A^\natural})$, and if $r \in \mathbb{Z}$ denotes the least integer with $\mathcal{H}^r E \neq 0$, then $\text{codim Supp } \mathcal{H}^r E = 2r$.

One application of Theorem 3.10 is to describe simple holonomic \mathcal{D}_A -modules with Euler characteristic zero. Recall that the *Euler characteristic* of a coherent algebraic \mathcal{D}_A -module \mathcal{M} is the integer

$$\chi(A, \mathcal{M}) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathbf{H}^k(A, \text{DR}_A(\mathcal{M})).$$

When \mathcal{M} is holonomic, we have $\chi(A, \mathcal{M}) \geq 0$ as a consequence of Theorem 3.9. In the regular case, the following result has independently been proved by Weissauer [Wei12, Theorem 2].

Corollary 3.11. *Let \mathcal{M} be a simple holonomic \mathcal{D}_A -module with $\chi(A, \mathcal{M}) = 0$. Then there is a surjective morphism with connected fibers $f: A \rightarrow B$ to a lower-dimensional abelian variety, and a simple holonomic \mathcal{D}_B -module \mathcal{N} , such that*

$$\mathcal{M} \simeq f^* \mathcal{N} \otimes_{\mathcal{O}_A} (L, \nabla)$$

for a suitable point $(L, \nabla) \in A^\natural$. Moreover, we may assume that $\chi(B, \mathcal{N}) > 0$.

4. Acknowledgements. This work was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan, and by NSF grant DMS-1100606. I thank Mihnea Popa and Pierre Schapira for their comments about the paper, and Takuro Mochizuki, Kiyoshi Takeuchi, Giovanni Morando, and Kentaro Hori for useful discussions. I am also very grateful to my parents-in-law for their hospitality during the preparation of the manuscript.

B. THE FOURIER-MUKAI TRANSFORM

5. The case of \mathcal{D} -modules. In this section, we recall the definition of the generalized Fourier-Mukai transform, introduced by Rothstein [Rot96] and Laumon [Lau96]. It is defined in the algebraic category, and so we begin by explaining why A^\natural is a complex algebraic variety. As always, let A be a complex abelian variety. The moduli space A^\natural of algebraic line bundles with integrable connection on A naturally has the structure of a quasi-projective algebraic variety: on the dual abelian variety $\hat{A} = \text{Pic}^0(A)$, there is a canonical extension of vector bundles

$$(5.1) \quad 0 \rightarrow \hat{A} \times H^0(A, \Omega_A^1) \rightarrow E(A) \rightarrow \hat{A} \times \mathbb{C} \rightarrow 0,$$

and A^\natural is isomorphic to the preimage of $\hat{A} \times \{1\}$ inside of $E(A)$. The projection

$$\pi: A^\natural \rightarrow \hat{A}, \quad (L, \nabla) \mapsto L,$$

is thus a torsor for the trivial bundle $\hat{A} \times H^0(A, \Omega_A^1)$; this corresponds to the fact that $\nabla + \omega$ is again an integrable connection for any $\omega \in H^0(A, \Omega_A^1)$. Note that A^\natural is a group under tensor product, with unit element the trivial line bundle (\mathcal{O}_A, d) .

The generalized Fourier-Mukai transform takes bounded complexes of coherent algebraic \mathcal{D}_A -modules to bounded complexes of algebraic coherent sheaves on A^\natural ; we briefly describe it following the presentation in [Lau96, §3]. Let P denote the normalized Poincaré bundle on the product $A \times \hat{A}$. Since A^\natural is the moduli space of line bundles with integrable connection on A , the pullback $P^\natural = (\text{id}_A \times \pi)^* P$ of the Poincaré bundle to the product $A \times A^\natural$ is endowed with a universal integrable connection

$$\nabla^\natural: P^\natural \rightarrow \Omega_{A \times A^\natural / A^\natural}^1 \otimes P^\natural$$

relative to A^\natural . Given any algebraic left \mathcal{D}_A -module \mathcal{M} , interpreted as a quasi-coherent sheaf of \mathcal{O}_A -modules with integrable connection $\nabla: \mathcal{M} \rightarrow \Omega_A^1 \otimes \mathcal{M}$, the tensor product $p_1^* \mathcal{M} \otimes_{\mathcal{O}_A} (P^\natural, \nabla^\natural)$ inherits a natural integrable connection relative to A^\natural . We then define the *Fourier-Mukai transform* of \mathcal{M} by the formula

$$(5.2) \quad \text{FM}_A(\mathcal{M}) = \mathbf{R}(p_2)_* \text{DR}_{A \times A^\natural / A^\natural}(p_1^* \mathcal{M} \otimes (P^\natural, \nabla^\natural))$$

where the relative de Rham complex

$$\left[p_1^* \mathcal{M} \otimes P^\natural \rightarrow \Omega_{A \times A^\natural / A^\natural}^1 \otimes p_1^* \mathcal{M} \otimes P^\natural \rightarrow \cdots \rightarrow \Omega_{A \times A^\natural / A^\natural}^g \otimes p_1^* \mathcal{M} \otimes P^\natural \right]$$

is placed in degrees $-g, \dots, 0$ as usual. Since every differential in the complex is \mathcal{O}_{A^\natural} -linear, it follows that $\text{FM}_A(\mathcal{M})$ is a complex of algebraic quasi-coherent sheaves on A^\natural . Finally, Laumon [Lau96, Théorème 3.2.1 and Corollaire 3.2.5] proves that this operation induces an equivalence

$$(5.3) \quad \text{FM}_A: \text{D}_{coh}^b(\mathcal{D}_A) \rightarrow \text{D}_{coh}^b(\mathcal{O}_{A^\natural})$$

between the bounded derived category of coherent algebraic \mathcal{D}_A -modules and the bounded derived category of algebraic coherent sheaves on A^\natural . Rothstein obtained the same result by a different method in [Rot96, Theorem 6.2].

Note. Since A is a complex projective variety, the category of coherent analytic \mathcal{D} -modules on A is equivalent to the category of coherent algebraic \mathcal{D} -modules by a version of the GAGA theorem. On the other hand, it is essential to consider only *algebraic* coherent sheaves on A^\natural in (5.3), because A^\natural is not projective.

We list some basic properties of the Fourier-Mukai transform. For $f: A \rightarrow B$ a surjective morphism of abelian varieties, one has the *direct image functor*

$$f_+: D_{coh}^b(\mathcal{D}_A) \rightarrow D_{coh}^b(\mathcal{D}_B), \quad f_+ \mathcal{M} = \mathbf{R}f_* \mathrm{DR}_{A/B}(\mathcal{M}),$$

where $\mathrm{DR}_{A/B}(\mathcal{M})$ denotes the relative de Rham complex

$$\mathrm{DR}_{A/B}(\mathcal{M}) = \left[\mathcal{M} \rightarrow \Omega_{A/B}^1 \otimes \mathcal{M} \rightarrow \cdots \rightarrow \Omega_{A/B}^r \otimes \mathcal{M} \right],$$

placed in degrees $-r, \dots, 0$, and $r = \dim A - \dim B$ is the relative dimension of f . For holonomic complexes, we have an induced functor

$$f_+: D_h^b(\mathcal{D}_A) \rightarrow D_h^b(\mathcal{D}_B)$$

since direct images under algebraic morphisms preserve holonomicity [HTT08, Theorem 3.2.3]. We also use the *shifted inverse image functor*

$$f^+ = \mathbf{L}f^*[\dim A - \dim B]: D_c^b(\mathcal{D}_B) \rightarrow D_c^b(\mathcal{D}_A),$$

which again preserves holonomic complexes since f is smooth. Finally, we use the *duality functor*

$$\mathbf{D}_A: D_{coh}^b(\mathcal{D}_A) \rightarrow D_{coh}^b(\mathcal{D}_A)^{opp}, \quad \mathbf{D}_A(\mathcal{M}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_A}(\mathcal{M}, (\Omega_A^g)^{-1} \otimes \mathcal{D}_A)[g].$$

Note that a \mathcal{D}_A -module \mathcal{M} is holonomic exactly when $\mathbf{D}_A(\mathcal{M})$ is again a \mathcal{D}_A -module, viewed as a complex concentrated in degree zero.

Theorem 5.4 (Laumon). *The Fourier-Mukai transform for \mathcal{D} -modules on abelian varieties has the following properties.*

- (a) For $(L, \nabla) \in A^\natural$, denote by $t_{(L, \nabla)}: A^\natural \rightarrow A^\natural$ the translation morphism. Then one has a canonical and functorial isomorphism

$$\mathrm{FM}_A(- \otimes_{\mathcal{O}_A} (L, \nabla)) = \mathbf{L}(t_{(L, \nabla)})^* \circ \mathrm{FM}_A.$$

- (b) One has a canonical and functorial isomorphism

$$\mathrm{FM}_A \circ \mathbf{D}_A = \langle -1_{A^\natural} \rangle^* \mathbf{R}\mathcal{H}om(\mathrm{FM}_A(-), \mathcal{O}_{A^\natural}).$$

- (c) For a surjective morphism $f: A \rightarrow B$, denote by $f^\natural: B^\natural \rightarrow A^\natural$ the induced morphism. Then one has a canonical and functorial isomorphism

$$\mathbf{L}(f^\natural)^* \circ \mathrm{FM}_A = \mathrm{FM}_B \circ f_+.$$

- (d) In the same situation, one has a canonical and functorial isomorphism

$$\mathbf{R}f_*^\natural \circ \mathrm{FM}_B = \mathrm{FM}_A \circ f^+.$$

Proof. (a) follows immediately from the properties of the Poincaré bundle on $A \times A^\natural$. (c) and (d) are proved in [Lau96, Proposition 3.3.2]; note that “ $g - 1 - g_2$ ” should read “ $g_1 - g_2$.” Lastly, (b) is contained in [Lau96, Proposition 3.3.4]. \square

6. The Rees algebra. Let A be a complex abelian variety of dimension g . The sheaf \mathcal{D}_A of linear differential operators is naturally filtered by the order of differential operators, and we consider the associated Rees algebra

$$\mathcal{R}_A = R_F \mathcal{D}_A = \bigoplus_{k=0}^{\infty} F_k \mathcal{D}_A \cdot z^k \subseteq \mathcal{D}_A \otimes_{\mathcal{O}_A} \mathcal{O}_A[z].$$

More concretely, let $\partial_1, \dots, \partial_g \in H^0(A, \mathcal{T}_A)$ be a basis for the space of tangent vector fields on A ; then as a sheaf of algebras, \mathcal{R}_A is generated over $\mathcal{O}_A[z]$ by the operators $\delta_1, \dots, \delta_g$, where $\delta_i = z\partial_i$, subject to the relations

$$[\delta_i, \delta_j] = 0 \quad \text{and} \quad [\delta_i, f] = z \cdot \partial_i f.$$

It is easy to see that we have $\mathcal{R}_A/(z-1)\mathcal{R}_A \simeq \mathcal{D}_A$, and $\mathcal{R}_A/z\mathcal{R}_A \simeq \text{Sym } \mathcal{T}_A$.

Definition 6.1. An algebraic \mathcal{R}_A -module is a sheaf of left \mathcal{R}_A -modules that is quasi-coherent over \mathcal{O}_A . An \mathcal{R}_A -module is called *strict* if it has no z -torsion.

Example 6.2. Let (\mathcal{M}, F) be a filtered \mathcal{D}_A -module; then the Rees module

$$R_F \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} F_k \mathcal{M} \cdot z^k$$

is a strict \mathcal{R}_A -module; it is coherent over \mathcal{R}_A iff the filtration $F = F_{\bullet} \mathcal{M}$ is good.

An equivalent point of view is the following. On the product

$$A \times \mathbb{C} = A \times_{\text{Spec } \mathbb{C}} \text{Spec } \mathbb{C}[z],$$

consider the subsheaf of $\mathcal{D}_{A \times \mathbb{C}/\mathbb{C}}$ generated by $z\mathcal{T}_{A \times \mathbb{C}/\mathbb{C}}$. For any quasi-coherent sheaf of $\mathcal{O}_{A \times \mathbb{C}}$ -modules with a left action by that sheaf of operators, the pushforward to A is then naturally an \mathcal{R}_A -module. Conversely, any algebraic \mathcal{R}_A -module \mathcal{M} gives rise to a quasi-coherent sheaf $\widetilde{\mathcal{M}}$ on $A \times \mathbb{C}$ that has the structure of a left module over the above sheaf of operators.

Given an \mathcal{R}_A -module \mathcal{M} , we have a $\mathbb{C}[z]$ -linear morphism of sheaves

$$\nabla: \widetilde{\mathcal{M}} \rightarrow \frac{1}{z} \Omega_{A \times \mathbb{C}/\mathbb{C}}^1 \otimes_{\mathcal{O}_{A \times \mathbb{C}}} \widetilde{\mathcal{M}}, \quad \nabla m = \sum_{i=1}^g \frac{\omega_i}{z} \otimes \delta_i m,$$

where $\omega_1, \dots, \omega_g \in H^0(A, \Omega_A^1)$ is the basis dual to $\partial_1, \dots, \partial_g \in H^0(A, \mathcal{T}_A)$. The *de Rham complex* of \mathcal{M} is the resulting complex

$$(6.3) \quad \text{DR}_A(\mathcal{M}) = \left[\widetilde{\mathcal{M}} \rightarrow \frac{1}{z} \Omega_{A \times \mathbb{C}/\mathbb{C}}^1 \otimes \widetilde{\mathcal{M}} \rightarrow \dots \rightarrow \frac{1}{z^g} \Omega_{A \times \mathbb{C}/\mathbb{C}}^g \otimes \widetilde{\mathcal{M}}, \right]$$

placed in degrees $-g, \dots, 0$, whose differentials are given by the formula

$$\frac{\omega}{z^k} \otimes m \mapsto (-1)^g \frac{d\omega}{z^{k+1}} \otimes zm + (-1)^{g+k} \frac{\omega}{z^k} \wedge \nabla m.$$

7. The moduli space of generalized connections. In this section, we introduce the moduli space of generalized connections on A ; it will be used in the following section to define a Fourier-Mukai transform for algebraic \mathcal{R}_A -modules. As explained in [Bon10], the idea of this construction is originally due to Deligne and Simpson.

Definition 7.1. Let X be a complex manifold, and $\lambda: X \rightarrow \mathbb{C}$ a holomorphic function. A *generalized connection with parameter λ* , or more briefly a λ -connection, on a locally free sheaf of \mathcal{O}_X -modules \mathcal{E} is a \mathbb{C} -linear morphism of sheaves

$$\nabla: \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$$

that satisfies the Leibniz rule with parameter λ , which is to say that

$$\nabla(f \cdot s) = f \cdot \nabla s + df \otimes \lambda s$$

for local sections $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$. A λ -connection is called *integrable* if its \mathcal{O}_X -linear curvature operator $\nabla \circ \nabla: \mathcal{E} \rightarrow \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$ is equal to zero.

Example 7.2. An integrable 1-connection is an integrable connection in the usual sense; an integrable 0-connection is the same thing as the structure of a Higgs bundle on \mathcal{E} .

On an abelian variety A , the moduli space $E(A)$ of line bundles with integrable λ -connection (for arbitrary $\lambda \in \mathbb{C}$) may be constructed as follows. Observe first that a λ -connection on a line bundle $L \in \text{Pic}(A)$ is integrable iff $L \in \text{Pic}^0(A)$. To construct the moduli space, let $\mathfrak{m}_A \subseteq \mathcal{O}_A$ denote the ideal sheaf of the unit element $0_A \in A$. Restriction of differential forms induces an isomorphism

$$\mathfrak{m}_A/\mathfrak{m}_A^2 = H^0(A, \Omega_A^1) \otimes \mathcal{O}_A/\mathfrak{m}_A,$$

and therefore determines an extension of coherent sheaves

$$0 \rightarrow H^0(A, \Omega_A^1) \otimes \mathcal{O}_A/\mathfrak{m}_A \rightarrow \mathcal{O}_A/\mathfrak{m}_A^2 \rightarrow \mathcal{O}_A/\mathfrak{m}_A \rightarrow 0.$$

Let P be the normalized Poincaré bundle on the product $A \times \widehat{A}$, and denote by $\mathbf{R}\Phi_P: D_{\text{coh}}^b(\mathcal{O}_A) \rightarrow D_{\text{coh}}^b(\mathcal{O}_{\widehat{A}})$ the Fourier-Mukai transform. Then $\mathbf{R}\Phi_P(\mathcal{O}_A/\mathfrak{m}_A^2)$ is a locally free sheaf $\mathcal{E}(A)$, and so we obtain an extension of locally free sheaves

$$(7.3) \quad 0 \rightarrow H^0(A, \Omega_A^1) \otimes \mathcal{O}_{\widehat{A}} \rightarrow \mathcal{E}(A) \rightarrow \mathcal{O}_{\widehat{A}} \rightarrow 0$$

on the dual abelian variety \widehat{A} . The resulting vector bundle extension

$$0 \rightarrow \widehat{A} \times H^0(A, \Omega_A^1) \rightarrow E(A) \rightarrow \widehat{A} \times \mathbb{C} \rightarrow 0$$

defines an algebraic vector bundle $E(A)$ on \widehat{A} ; by construction, it comes with two algebraic morphisms $\pi: E(A) \rightarrow \widehat{A}$ and $\lambda: E(A) \rightarrow \mathbb{C}$.

Now the claim is that $E(A)$ is the moduli space of line bundles with integrable λ -connection, with $\pi: E(A) \rightarrow \widehat{A}$ the map that takes (L, ∇, λ) to the underlying line bundle L . The following lemma establishes the existence of a tautological line bundle with generalized connection on $A \times E(A)$.

Lemma 7.4. *Let $\tilde{P} = (\text{id} \times \pi)^* P$ denote the pullback of the Poincaré bundle to $A \times E(A)$. Then there is a canonical generalized relative connection*

$$\tilde{\nabla}: \tilde{P} \rightarrow \Omega_{A \times E(A)/E(A)}^1 \otimes \tilde{P}$$

that satisfies the Leibniz rule $\tilde{\nabla}(f \cdot s) = f \cdot \tilde{\nabla}s + d_{A \times E(A)/E(A)} f \otimes \lambda s$.

Proof. Let \mathcal{I} denote the ideal sheaf of the diagonal in $A \times A$. Let Z be the non-reduced subscheme of $A \times A \times E(A)$ defined by the ideal sheaf $\mathcal{I}^2 \cdot \mathcal{O}_{A \times A \times E(A)}$. We have a natural exact sequence

$$(7.5) \quad 0 \rightarrow \tilde{P} \otimes H^0(A, \Omega_A^1) \rightarrow (p_{13})_*(\mathcal{O}_Z \otimes p_{23}^* \tilde{P}) \rightarrow \tilde{P} \rightarrow 0,$$

and a generalized relative connection is the same thing as a morphism of sheaves

$$\tilde{P} \rightarrow (p_{13})_*(\mathcal{O}_Z \otimes p_{23}^* \tilde{P})$$

whose composition with the morphism to \tilde{P} acts as multiplication by λ . In fact, there is a canonical choice, which we shall now describe. Consider the morphism

$$f: A \times A \rightarrow A \times A, \quad f(a, b) = (a, a + b).$$

Since $f \times \text{id}_{E(A)}$ induces an isomorphism between the first infinitesimal neighborhood of $A \times \{0_A\} \times E(A)$ and the subscheme Z , we have

$$\begin{aligned} (f \times \text{id}_{E(A)})^*(\mathcal{O}_Z \otimes p_{23}^* \tilde{P}) &= p_2^*(\mathcal{O}_A/\mathfrak{m}_A^2) \otimes (m \times \text{id}_{E(A)})^* \tilde{P} \\ &= p_2^*(\mathcal{O}_A/\mathfrak{m}_A^2) \otimes p_{13}^* \tilde{P} \otimes p_{23}^* \tilde{P}, \end{aligned}$$

due to the well-known fact that the Poincaré bundle satisfies

$$(m \times \text{id}_{\hat{A}})^* P = p_{13}^* P \otimes p_{23}^* P$$

on $A \times A \times \hat{A}$. Since $p_{13} \circ (f \times \text{id}_{E(A)}) = p_{13}$, we conclude that we have

$$(p_{13})_*(\mathcal{O}_Z \otimes p_{23}^* \tilde{P}) = \tilde{P} \otimes p_2^* \pi^* \mathbf{R}\Phi_P(\mathcal{O}_A/\mathfrak{m}_A^2) = \tilde{P} \otimes p_2^* \pi^* \mathcal{E}(A)$$

on $A \times E(A)$; more precisely, (7.5) is isomorphic to the tensor product of \tilde{P} and the pullback of (7.3) by $\pi \circ p_2$.

Now the pullback of the exact sequence (7.3) to $E(A)$ obviously has a splitting of the type we are looking for: indeed, the tautological section of $\pi^* \mathcal{E}(A)$ gives a morphism $\mathcal{O}_{E(A)} \rightarrow \pi^* \mathcal{E}(A)$ whose composition with the projection to $\mathcal{O}_{E(A)}$ is multiplication by λ . Thus we obtain a canonical morphism $\tilde{P} \rightarrow \tilde{P} \otimes p_2^* \pi^* \mathcal{E}(A)$ and hence, by the above, the desired generalized relative connection. \square

Note. At any closed point $e \in E(A)$, we obtain a $\lambda(e)$ -connection on the line bundle corresponding to $\pi(e) \in \text{Pic}^0(A)$. Using the properties of the Picard scheme, it is not difficult to show that $E(A)$ is a fine moduli space, in the obvious sense; but since we do not need this fact below, we shall omit the proof.

Corollary 7.6. *The pullback of the Poincaré bundle to $A \times E(A)$ has the structure of a relative \mathcal{R}_A -module, where z act as multiplication by λ , and δ_i as ∇_{∂_i} .*

8. The case of \mathcal{R} -modules. We shall now describe a version of the Fourier-Mukai transform that works for algebraic \mathcal{R}_A -modules. Since we only need a very special case in this paper, we leave a more careful discussion to a future publication.

Let \mathcal{M} be an \mathcal{R}_A -module, and denote by $\tilde{\mathcal{M}}$ the associated quasi-coherent sheaf on $A \times \mathbb{C}$; as explained above, $\tilde{\mathcal{M}}$ is a left module over the subalgebra of $\mathcal{D}_{A \times \mathbb{C}/\mathbb{C}}$ generated by $z\mathcal{T}_{A \times \mathbb{C}/\mathbb{C}}$. Thus the tensor product

$$(\text{id} \times \lambda)^* \tilde{\mathcal{M}} \otimes_{\mathcal{O}_{A \times E(A)}} \tilde{P}$$

naturally has the structure of a relative \mathcal{R}_A -module on $A \times E(A)$, with $z(m \otimes s) = (\lambda m) \otimes s = m \otimes \lambda s$ and $\delta_i(m \otimes s) = (\delta_i m) \otimes s + m \otimes \tilde{\nabla}_{\partial_i} s$. We may therefore consider the relative de Rham complex

$$\text{DR}_{A \times E(A)/E(A)} \left((\text{id} \times \lambda)^* \tilde{\mathcal{M}} \otimes_{\mathcal{O}_{A \times E(A)}} (\tilde{P}, \tilde{\nabla}) \right),$$

which is defined analogously to (6.3).

Definition 8.1. The *Fourier-Mukai transform* of an \mathcal{R}_A -module \mathcal{M} is

$$\widetilde{\text{FM}}_A(\mathcal{M}) = \mathbf{R}(p_2)_* \text{DR}_{A \times E(A)/E(A)} \left((\text{id} \times \lambda)^* \tilde{\mathcal{M}} \otimes_{\mathcal{O}_{A \times E(A)}} \tilde{P} \right);$$

it is an object of $\text{D}^b(\mathcal{O}_{E(A)})$, the bounded derived category of quasi-coherent sheaves on $E(A)$.

Note. Using the general formalism in [PR01], one can show that the Fourier-Mukai transform induces an equivalence of categories

$$\widetilde{\text{FM}}_A: \text{D}^b(\mathcal{R}_A) \rightarrow \text{D}^b(\mathcal{O}_{E(A)}),$$

which restricts to an equivalence between the coherent subcategories. Since this fact will not be used below, we again omit the proof.

9. Compatibility. Just as \mathcal{R}_A -modules interpolate between \mathcal{D}_A -modules and quasi-coherent sheaves on the cotangent bundle T^*A , Definition 8.1 interpolates between the Fourier-Mukai transform for \mathcal{D}_A -modules and the usual Fourier-Mukai transform for quasi-coherent sheaves. The purpose of this section is to make that relationship precise.

Throughout the discussion, let \mathcal{M} be a coherent \mathcal{D}_A -module and $F = F_\bullet \mathcal{M}$ a good filtration of \mathcal{M} by \mathcal{O}_A -coherent subsheaves. The graded $\text{Sym } \mathcal{T}_A$ -module $\text{gr}_\bullet^F \mathcal{M}$ is then coherent over $\text{Sym } \mathcal{T}_A$, and therefore defines a coherent sheaf on the cotangent bundle T^*A that we denote by $\mathcal{C}(\mathcal{M}, F)$. Now consider the Rees module

$$R_F \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} F_k \mathcal{M} \cdot z^k \subseteq \mathcal{M} \otimes_{\mathcal{O}_A} \mathcal{O}_A[z, z^{-1}],$$

which is a graded \mathcal{R}_A -module, coherent over \mathcal{R}_A . The associated quasi-coherent sheaf on $A \times \mathbb{C}$, which we again denote by the symbol $\widetilde{R_F \mathcal{M}}$, is then equivariant for the natural \mathbb{C}^* -action on the product. Moreover, it is easy to see that the restriction of $\widetilde{R_F \mathcal{M}}$ to $A \times \{1\}$ is a \mathcal{D}_A -module isomorphic to \mathcal{M} , while the restriction to $A \times \{0\}$ is a $\text{Sym } \mathcal{T}_A$ -module isomorphic to $\text{gr}_\bullet^F \mathcal{M}$.

Proposition 9.1. *Let \mathcal{M} be a coherent \mathcal{D}_A -module with good filtration $F_\bullet \mathcal{M}$. Then the Fourier-Mukai transform $\widetilde{\text{FM}}_A(R_F \mathcal{M})$ has the following properties:*

- (i) *It is equivariant for the natural \mathbb{C}^* -action on the vector bundle $E(A)$.*
- (ii) *Its restriction to $A^\natural = \lambda^{-1}(1)$ is canonically isomorphic to $\text{FM}_A(\mathcal{M})$.*
- (iii) *Its restriction to $A \times H^0(A, \Omega_A^1) = \lambda^{-1}(0)$ is canonically isomorphic to*

$$\mathbf{R}(p_{23})_* \left(p_{12}^* P \otimes p_1^* \Omega_A^g \otimes p_{13}^* \mathcal{C}(\mathcal{M}, F) \right),$$

where the notation is as in the diagram in (9.3) below.

We begin the proof with two simple lemmas that describe how $E(A)$ and $(\tilde{P}, \tilde{\nabla})$ behave under restriction to the fibers of $\lambda: E(A) \rightarrow \mathbb{C}$.

Lemma 9.2. *We have $\lambda^{-1}(1) = A^\natural$, and the restriction of $(\tilde{P}, \tilde{\nabla})$ to $A \times A^\natural$ is equal to $(P^\natural, \nabla^\natural)$.*

Proof. This follows from the construction of A^\natural in [MM74, Chapter I]. \square

Recall that the cotangent bundle of A satisfies $T^*A = A \times H^0(A, \Omega_A^1)$, and consider the following diagram:

$$(9.3) \quad \begin{array}{ccc} A \times H^0(A, \Omega_A^1) & \xleftarrow{p_{13}} & A \times \hat{A} \times H^0(A, \Omega_A^1) \xrightarrow{p_{23}} \hat{A} \times H^0(A, \Omega_A^1) \\ & & \downarrow p_{12} \\ & & A \times \hat{A} \end{array}$$

Lemma 9.4. *We have $\lambda^{-1}(0) = \hat{A} \times H^0(A, \Omega_A^1)$, and the restriction of $(\tilde{P}, \tilde{\nabla})$ to $A \times \hat{A} \times H^0(A, \Omega_A^1)$ is equal to the Higgs bundle*

$$(p_{12}^* P, p_{23}^* \theta_A),$$

where θ_A denotes the tautological holomorphic one-form on T^*A .

Proof. This follows easily from the proof of Lemma 7.4. \square

We can now prove the asserted compatibilities between Definition 8.1 and the Fourier-Mukai transforms for \mathcal{D}_A -modules and coherent sheaves.

Proof of Proposition 9.1. (i) is true because $R_F \mathcal{M}$ is a graded \mathcal{R}_A -module, and because \tilde{P} , $\tilde{\nabla}$, and the relative de Rham complex are obviously \mathbb{C}^* -equivariant. (ii) follows directly from the definition of the Fourier-Mukai transform, using the base change formula for the morphism $\lambda: E(A) \rightarrow \mathbb{C}$ and Lemma 9.2.

The proof of (iii) is a little less obvious, and so we give some details. By base change, it suffices to show that the restriction of the relative de Rham complex

$$\mathrm{DR}_{A \times E(A)/E(A)} \left((\mathrm{id} \times \lambda)^* \tilde{\mathcal{M}} \otimes_{\mathcal{O}_{A \times E(A)}} (\tilde{P}, \tilde{\nabla}) \right)$$

to $A \times \hat{A} \times H^0(A, \Omega_A^1)$ is a resolution of the coherent sheaf $p_{12}^* P \otimes p_1^* \Omega_A^g \otimes p_{13}^* \mathcal{C}(\mathcal{M}, F)$. After a short computation, one finds that this restriction is isomorphic to the tensor product of $p_{12}^* P$ and the pullback, via $p_{13}: A \times \hat{A} \times H^0(A, \Omega_A^1) \rightarrow A \times H^0(A, \Omega_A^1)$, of the complex

$$\left[p_1^* (\mathrm{gr}_{\bullet}^F \mathcal{M}) \rightarrow p_1^* (\Omega_A^1 \otimes_{\mathcal{O}_A} \mathrm{gr}_{\bullet}^F \mathcal{M}) \rightarrow \cdots \rightarrow p_1^* (\Omega_A^g \otimes_{\mathcal{O}_A} \mathrm{gr}_{\bullet}^F \mathcal{M}) \right],$$

placed in degrees $-g, \dots, 0$, with differential $p_1^* (\Omega_A^k \otimes \mathrm{gr}_{\bullet}^F \mathcal{M}) \rightarrow p_1^* (\Omega_A^{k+1} \otimes \mathrm{gr}_{\bullet}^F \mathcal{M})$ given by the formula

$$\omega \otimes m \mapsto (-1)^{g+k} \left(\omega \wedge \theta_A \otimes m + \sum_{i=1}^g \omega \wedge \omega_i \otimes \partial_i m \right).$$

But since $\mathcal{C}(\mathcal{M}, F)$ is the coherent sheaf on $A \times H^0(A, \Omega_A^1)$ corresponding to the Sym \mathcal{R}_A -module $\mathrm{gr}_{\bullet}^F \mathcal{M}$, said complex resolves the coherent sheaf $p_1^* \Omega_A^g \otimes \mathcal{C}(\mathcal{M}, F)$, and so we get the desired result. \square

C. RESULTS ABOUT COHOMOLOGY SUPPORT LOCI

10. Cohomology of constructible complexes. In this section, we describe an analogue of the Fourier-Mukai transform for constructible complexes on A , and use it to prove that cohomology support loci are algebraic subvarieties of $\mathrm{Char}(A)$. We refer the reader to [HTT08, Section 4.5] and to [Dim04, Chapter 4] for details about constructible complexes and perverse sheaves.

As a complex manifold, the abelian variety A may be presented as a quotient V/Λ , where V is a complex vector space of dimension g , and $\Lambda \subseteq V$ is a lattice of rank $2g$. Note that V is isomorphic to the tangent space of A at the unit element, while Λ is isomorphic to the fundamental group $\pi_1(A, 0_A)$. We shall denote by $R = \mathbb{C}[\Lambda]$ the group ring of Λ ; thus

$$R = \bigoplus_{\lambda \in \Lambda} \mathbb{C} e_{\lambda},$$

with $e_{\lambda} \cdot e_{\mu} = e_{\lambda+\mu}$. A choice of basis for Λ shows that R is isomorphic to the ring of Laurent polynomials in $2g$ variables. Any character $\rho: \Lambda \rightarrow \mathbb{C}^*$ extends uniquely to a homomorphism of \mathbb{C} -algebras

$$R \rightarrow \mathbb{C}, \quad e_{\lambda} \mapsto \rho(\lambda),$$

whose kernel is a maximal ideal \mathfrak{m}_{ρ} of R ; concretely, \mathfrak{m}_{ρ} is generated by the elements $e_{\lambda} - \rho(\lambda)$ for $\lambda \in \Lambda$. It is easy to see that any maximal ideal of R is of this form; this means that $\mathrm{Char}(A)$ is the set of closed points of the scheme $\mathrm{Spec} R$, and therefore naturally an affine algebraic variety over $\mathrm{Spec} \mathbb{C}$.

For any finitely generated R -module M , multiplication by the ring elements e_{λ} determines a natural action of Λ on the \mathbb{C} -vector space M . By the well-known correspondence between representations of the fundamental group and local systems, it thus gives rise to a local system on A .

Definition 10.1. For a finitely generated R -module M , we denote by the symbol \mathcal{L}_M the corresponding local system of \mathbb{C} -vector spaces on A .

Since R is commutative, \mathcal{L}_M is actually a *local system of R -modules*. The most important case of this construction is \mathcal{L}_R , which is a local system of R -modules of rank one; one can show that it is isomorphic to the direct image with proper support $\pi_! \mathbb{C}_V$ of the constant local system on the covering space $\pi: V \rightarrow A$, but we do not need this fact here. The device above allows us to construct finitely-generated R -modules (and hence coherent sheaves on $\text{Spec } R$) by twisting a constructible complex on A by a local system of the form \mathcal{L}_M , and pushing forward along the morphism $p: A \rightarrow \text{Spec } \mathbb{C}$ to a point.

Proposition 10.2. *Let $E \in D_c^b(\mathbb{C}_A)$. Then for any finitely generated R -module M , the direct image $\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_M)$ belongs to $D_{coh}^b(R)$.*

Proof. Since E is a constructible complex of sheaves of \mathbb{C} -vector spaces, the tensor product $E \otimes_{\mathbb{C}} \mathcal{L}_M$ is a constructible complex of sheaves of R -modules. By [Dim04, Corollary 4.1.6], its direct image is thus an object of $D_{coh}^b(R)$. \square

To understand how $\mathbf{R}p_*(E \otimes_{\mathcal{O}_A} \mathcal{L}_M)$ depends on M , we will need the following auxiliary lemma. Recall that a *fine sheaf* on a manifold is a sheaf admitting partitions of unity; such sheaves are acyclic for direct image functors.

Lemma 10.3. *Let \mathcal{F} be a fine sheaf of \mathbb{C} -vector spaces on A . Then the space of global sections $H^0(A, \mathcal{F} \otimes_{\mathbb{C}} \mathcal{L}_R)$ is a flat R -module, and for every finitely generated R -module M , one has*

$$H^0(A, \mathcal{F} \otimes_{\mathbb{C}} \mathcal{L}_M) \simeq H^0(A, \mathcal{F} \otimes_{\mathbb{C}} \mathcal{L}_R) \otimes_R M,$$

functorially in M .

Proof. It is easy to see that each sheaf of the form $\mathcal{F} \otimes_{\mathbb{C}} \mathcal{L}_M$ is again a fine sheaf. Consequently, $M \mapsto H^0(A, \mathcal{F} \otimes_{\mathbb{C}} \mathcal{L}_M)$ is an exact functor from the category of finitely generated R -modules to the category of R -modules. Since the functor also preserves direct sums, the result follows from the Eilenberg-Watts theorem in homological algebra [Wat60]. A direct proof can be had by resolving M by a bounded complex of free R -modules of finite rank, and using the exactness of the functor. \square

Proposition 10.4. *Let $E \in D_c^b(\mathbb{C}_A)$. Then for every finitely generated R -module M , one has an isomorphism*

$$\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_M) \simeq \mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_R) \overset{\mathbf{L}}{\otimes}_R M,$$

functorial in M .

Proof. We begin by choosing a bounded complex (\mathcal{F}^\bullet, d) of fine sheaves quasi-isomorphic to E . One way to do this is as follows. By the Riemann-Hilbert correspondence, $E \simeq \text{DR}_A(\mathcal{M})$ for some $\mathcal{M} \in D_{rh}^b(\mathcal{D}_A)$; if we now let \mathcal{A}_A^k denote the sheaf of smooth k -forms on the complex manifold A , then by the Poincaré lemma,

$$\left[\mathcal{A}_A^0 \otimes \mathcal{M} \rightarrow \mathcal{A}_A^1 \otimes \mathcal{M} \rightarrow \cdots \rightarrow \mathcal{A}_A^{2g} \otimes \mathcal{M} \right],$$

placed in degrees $-g, \dots, g$, is a complex of fine sheaves quasi-isomorphic to E . For any such choice, $\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_M) \in D_{coh}^b(R)$ is represented by the bounded complex of R -modules with terms

$$H^0(A, \mathcal{F}^\bullet \otimes_{\mathbb{C}} \mathcal{L}_M) \simeq H^0(A, \mathcal{F}^\bullet \otimes_{\mathbb{C}} \mathcal{L}_R) \otimes_R M,$$

and so the assertion follows from Lemma 10.3. \square

Now let $\rho \in \text{Char}(A)$ be an arbitrary character; recall that \mathfrak{m}_ρ is the maximal ideal of R generated by the elements $e_\lambda - \rho(\lambda)$, for $\lambda \in \Lambda$. Using the notation introduced above, we therefore have the alternative description $\mathbb{C}_\rho \simeq \mathcal{L}_{R/\mathfrak{m}_\rho}$ for the local system corresponding to ρ .

Corollary 10.5. *For any character $\rho \in \text{Char}(A)$, we have*

$$\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathbb{C}_\rho) \simeq \mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_R) \overset{\mathbf{L}}{\otimes}_R R/\mathfrak{m}_\rho$$

as objects of $D_{coh}^b(\mathbb{C})$.

Note. We may thus consider $\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_R) \in D_{coh}^b(R)$ as being something like a “Fourier-Mukai transform” of the constructible complex $E \in D_c^b(\mathbb{C}_A)$. In this setting, however, the transform does not determine the original constructible complex: for example, if E is any constructible sheaf whose support is a finite union of points of A , then $\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_R)$ is a free R -module of finite rank.

The results above are sufficient to prove that the cohomology support loci of E are algebraic subsets of $\text{Char}(A)$.

Theorem 10.6. *If $E \in D_c^b(\mathbb{C}_A)$, then each cohomology support locus $S_m^k(E)$ is an algebraic subset of $\text{Char}(A)$.*

Proof. Recall that $\text{Char}(A)$ is the complex manifold associated to the complex algebraic variety $\text{Spec } R$. Thus $\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_R) \in D_{coh}^b(R)$ determines an object in the bounded derived category of algebraic coherent sheaves on $\text{Char}(A)$, whose fiber at any closed point ρ computes the hypercohomology of the twist $E \otimes_{\mathbb{C}} \mathbb{C}_\rho$, according to Corollary 10.5. We conclude that

$$S_m^k(E) = \left\{ \rho \in \text{Char}(A) \mid \dim \mathbf{H}^k\left(\mathbf{R}p_*(E \otimes_{\mathbb{C}} \mathcal{L}_R) \overset{\mathbf{L}}{\otimes}_R R/\mathfrak{m}_\rho\right) \geq m \right\},$$

and by the same argument as in the proof of Proposition 11.3, this description implies that $S_m^k(E)$ is an algebraic subset of $\text{Char}(A)$. \square

Proposition 10.7. *Let k be any subfield of \mathbb{C} . If $E \in D_c^b(k_A)$ is a constructible complex of sheaves of k -vector spaces, then $\mathbf{R}p_*(E \otimes_k \mathcal{L}_R)$ is defined over k .*

Proof. Let $\mathbb{Q}[\Lambda]$ be the group ring with rational coefficients. Then \mathcal{L}_R is the complexification of the associated local system of \mathbb{Q} -vector spaces, and so $\mathbf{R}p_*(E \otimes_k \mathcal{L}_R)$ is in an evident manner the complexification of an object of $D_{coh}^b(k[\Lambda])$. \square

11. Structure theorem. The goal of this section is to prove a fundamental structure theorem for cohomology support loci of constructible and holonomic complexes. We refer to [HTT08, Chapter 3] for details about holonomic \mathcal{D} -modules and holonomic complexes.

Lemma 11.1. *Let $\mathcal{M} \in D_h^b(\mathcal{D}_A)$ be a holonomic complex on A . Then we have*

$$\Phi(S_m^k(\mathcal{M})) = S_m^k(\text{DR}_A(\mathcal{M})).$$

for every $k, m \in \mathbb{Z}$.

Proof. Let (L, ∇) be a line bundle with integrable connection on A . The associated local system $\ker \nabla$ is a subsheaf of L , and so we obtain a morphism of complexes

$$(11.2) \quad \text{DR}_A(\mathcal{M}) \otimes_{\mathbb{C}} (\ker \nabla) \rightarrow \text{DR}_A(\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla)).$$

Now $(\ker \nabla) \otimes_{\mathbb{C}} \mathcal{O}_A = L$, and therefore $\Omega_A^k \otimes_{\mathcal{O}_A} \mathcal{M} \otimes_{\mathbb{C}} (\ker \nabla) = \Omega_A^k \otimes_{\mathcal{O}_A} \mathcal{M} \otimes_{\mathcal{O}_A} L$. This shows that the two complexes in (11.2) are isomorphic to each other, and gives the desired relation between their hypercohomology groups. \square

With the help of the Fourier-Mukai transform, it is easy to show that cohomology support loci are algebraic subsets of A^\natural .

Proposition 11.3. *If $\mathcal{M} \in D_{coh}^b(\mathcal{D}_A)$, then each cohomology support locus $S_m^k(\mathcal{M})$ is an algebraic subset of A^\natural .*

Proof. Since A^\natural is a quasi-projective algebraic variety, we may represent $\mathrm{FM}_A(\mathcal{M})$ by a bounded complex (\mathcal{E}^\bullet, d) of locally free sheaves on A^\natural . Now let (L, ∇) be a line bundle with integrable connection, and let $i_{(L, \nabla)}$ denote the inclusion map. By the base change theorem,

$$\mathbf{R}i_{(L, \nabla)}^* \mathrm{FM}_A(\mathcal{M}) \simeq \mathrm{DR}_A(\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla)),$$

and so we have

$$S_m^k(\mathcal{M}) = \left\{ (L, \nabla) \in A^\natural \mid \dim \mathbf{H}^k(i_{(L, \nabla)}^*(\mathcal{E}^\bullet, d)) \geq m \right\}.$$

This description shows that $S_m^k(\mathcal{M})$ is an algebraic subset of A^\natural , as claimed. \square

We can now prove the structure theorem from the introduction.

Proof of Theorem 3.3. Let $\mathcal{M} \in D_h^b(\mathcal{D}_A)$ be a holonomic complex. Then $\mathrm{DR}_A(\mathcal{M})$ is constructible, and we have

$$\Phi(S_m^k(\mathcal{M})) = S_m^k(\mathrm{DR}_A(\mathcal{M}))$$

by Lemma 11.1. Proposition 11.3 shows that $S_m^k(\mathcal{M})$ is an algebraic subset of A^\natural ; Theorem 10.6 shows that $S_m^k(\mathrm{DR}_A(\mathcal{M}))$ is an algebraic subset of $\mathrm{Char}(A)$. We conclude from Simpson's Theorem 3.6 that both must be finite unions of linear subvarieties of A^\natural and $\mathrm{Char}(A)$, respectively. The assertion about objects of geometric origin is proved in §12 below. \square

Proof of Corollary 3.7. A familiar consequence of the base change theorem is that we have, for every $n \in \mathbb{Z}$, an equality of sets

$$(11.4) \quad \bigcup_{k \geq n} \mathrm{Supp} \mathcal{H}^k \mathrm{FM}_A(\mathcal{M}) = \bigcup_{k \geq n} S^k(\mathcal{M}).$$

Both assertions therefore follow from Theorem 3.3. \square

12. Objects of geometric origin. In this section, we study cohomology support loci for semisimple regular holonomic \mathcal{D}_A -modules of geometric origin, as defined in [BBD82, 6.2.4]. To begin with, recall the following definition due to Saito [Sai91, Definition 2.6].

Definition 12.1. A mixed Hodge module is said to be *of geometric origin* if it is obtained by applying several of the standard cohomological functors $H^i f_*$, $H^i f^!$, $H^i f^*$, $H^i f^!$, ψ_g , $\phi_{g,1}$, \mathbf{D} , \boxtimes , \oplus , \otimes , and $\mathcal{H}om$ to the trivial Hodge structure \mathbb{Q}^H of weight zero, and then taking subquotients in the category of mixed Hodge modules.

One of the results of Saito's theory is that any semisimple perverse sheaf of geometric origin, in the sense of [BBD82, 6.2.4], is a direct summand of a perverse sheaf underlying a mixed Hodge module of geometric origin. Consequently, any semisimple regular holonomic \mathcal{D} -module of geometric origin is a direct summand of a \mathcal{D} -module underlying a mixed Hodge module of geometric origin.

Theorem 12.2. *Let \mathcal{M} be a semisimple regular holonomic \mathcal{D}_A -module of geometric origin. Then each cohomology support locus $S_m^k(\mathcal{M})$ is a finite union of arithmetic linear subvarieties of A^\natural .*

We introduce some notation that will be used during the proof. For any field automorphism $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$, we obtain from A a new complex abelian variety A^σ . Likewise, an algebraic line bundle (L, ∇) with integrable connection on A gives rise to $(L^\sigma, \nabla^\sigma)$ on A^σ , and so we have a natural map

$$c_\sigma: A^\natural \rightarrow (A^\sigma)^\natural.$$

Now recall the following notion, due in a slightly different form to Simpson, who modeled it on Deligne's definition of absolute Hodge classes.

Definition 12.3. A closed subset $Z \subseteq A^\natural$ is said to be *absolute closed* if, for every field automorphism $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$, the set

$$F(c_\sigma(Z)) \in \text{Char}(A^\sigma)$$

is closed and defined over $\bar{\mathbb{Q}}$.

The following theorem about absolute closed subsets is also due to Simpson.

Theorem 12.4 (Simpson). *An absolute closed subset of A^\natural is a finite union of arithmetic linear subvarieties.*

Proof. Simpson's definition [Sim93, p. 376] of absolute closed sets actually contains several additional conditions (related to the space of Higgs bundles); but as he explains, a strengthening of [Sim93, Theorem 3.1], added in proof, makes these conditions unnecessary. In fact, the proof of [Sim93, Theorem 6.1] goes through unchanged with only the assumptions in Definition 12.3. \square

With the help of Simpson's result, the proof of Theorem 12.2 is straightforward. We first establish the following lemma.

Lemma 12.5. *Let $M \in \text{MHM}(A)$ be a mixed Hodge module, with underlying filtered \mathcal{D}_A -module (\mathcal{M}, F) . Then the cohomology support loci of the perverse sheaf $\text{DR}_A(\mathcal{M})$ are algebraic subsets of $\text{Char}(A)$ that are defined over $\bar{\mathbb{Q}}$.*

Proof. By definition, a mixed Hodge module has an underlying perverse sheaf $\text{rat } M$ with coefficients in \mathbb{Q} , and $\text{DR}_A(\mathcal{M}) \simeq (\text{rat } M) \otimes_{\mathbb{Q}} \mathbb{C}$. By Proposition 10.7, it follows that $\mathbf{R}p_*(\text{DR}_A(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{L}_R) \in \text{D}_{\text{coh}}^b(R)$ is obtained by extension of scalars from an object of $\text{D}_{\text{coh}}^b(\mathbb{Q}[\Lambda])$. The assertion about cohomology support loci now follows easily from Corollary 10.5. \square

Note. The same result is true for any holonomic \mathcal{D}_A -module with $\bar{\mathbb{Q}}$ -structure; that is to say, for any holonomic \mathcal{D}_A -module whose de Rham complex is the complexification of a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$. This is what Mochizuki calls a “pre-Betti structure” in [Moc10].

Lemma 12.6. *Let $E \in \text{D}_c^b(\bar{\mathbb{Q}}_A)$ be a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$. Any simple subquotient of $E \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ is the complexification of a simple subquotient of E .*

Proof. We only have to show that if $E \in \text{D}_c^b(\bar{\mathbb{Q}}_A)$ is a simple perverse sheaf, then $E \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \in \text{D}_c^b(\mathbb{C}_A)$ is also simple. By the classification of simple perverse sheaves, there is an irreducible locally closed subvariety $U \subseteq A$, and an irreducible representation $\rho: \pi_1(U) \rightarrow \text{GL}_n(\bar{\mathbb{Q}})$, such that E is the intermediate extension of the local system associated to ρ . Since $\bar{\mathbb{Q}}$ is algebraically closed, ρ remains irreducible over \mathbb{C} , proving that $E \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ is still simple. \square

Proof of Theorem 12.2. We first show that this holds when \mathcal{M} underlies a mixed Hodge module M obtained by iterating the standard cohomological functors (but without taking subquotients). Fix two integers k, m , and set $Z = S_m^k(\mathcal{M})$. In light of Lemma 12.5, it suffices to prove that each set $c_\sigma(Z)$ is equal to $S_m^k(\mathcal{M}_\sigma)$ for some

polarizable Hodge module $M_\sigma \in \text{MHM}(A^\sigma)$. But since M is of geometric origin, this is obviously the case; indeed, we can obtain M_σ by simply applying σ to the finitely many algebraic varieties and morphisms involved in the construction of M .

Now suppose that \mathcal{M} is an arbitrary semisimple regular holonomic \mathcal{D}_A -module of geometric origin. Then \mathcal{M} is a direct sum of simple subquotients of \mathcal{D}_A -modules underlying mixed Hodge modules of geometric origin. By the same argument as before, it suffices to show that \mathcal{M}_σ is defined over $\bar{\mathbb{Q}}$ for any $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$. Now the perverse sheaf $\text{DR}_A(\mathcal{M}_\sigma)$ is again a direct sum of simple subquotients of perverse sheaves underlying mixed Hodge modules; by Lemma 12.6, it is therefore the complexification of a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$. We then conclude the proof as above. \square

13. Perverse coherent sheaves. Let X be a smooth complex algebraic variety. In this section, we recall the construction of perverse t-structures on the bounded derived category $D_{\text{coh}}^b(\mathcal{O}_X)$ of algebraic coherent sheaves, following [Kas04]. For a (possibly non-closed) point x of the scheme X , we denote the residue field at the point by $\kappa(x)$, the inclusion morphism by $i_x: \text{Spec } \kappa(x) \hookrightarrow X$, and the codimension of the closed subvariety $\overline{\{x\}}$ by $\text{codim}(x) = \dim \mathcal{O}_{X,x}$.

Definition 13.1. A *supporting function* on X is a function $p: X \rightarrow \mathbb{Z}$ from the underlying topological space of the scheme X to the set of integers, with the property that $p(y) \geq p(x)$ whenever $y \in \overline{\{x\}}$.

Given a supporting function, Kashiwara defines two families of subcategories

$$\begin{aligned} {}^pD_{\text{coh}}^{\leq k}(\mathcal{O}_X) &= \{ E \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \mathbf{L}i_x^* E \in D_{\text{coh}}^{\leq k+p(x)}(\kappa(x)) \text{ for all } x \in X \}, \\ {}^pD_{\text{coh}}^{\geq k}(\mathcal{O}_X) &= \{ E \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \mathbf{R}i_x^! E \in D_{\text{coh}}^{\geq k+p(x)}(\kappa(x)) \text{ for all } x \in X \}. \end{aligned}$$

The following fundamental result is proved in [Kas04, Theorem 5.9] and, based on an idea of Deligne, in [AB10, Theorem 3.10].

Theorem 13.2 (Kashiwara). *The above subcategories define a bounded t-structure on $D_{\text{coh}}^b(\mathcal{O}_X)$ if, and only if, the supporting function has the property that*

$$p(y) - p(x) \leq \text{codim}(y) - \text{codim}(x)$$

for every pair of (possibly non-closed) points $x, y \in X$ with $y \in \overline{\{x\}}$.

For example, $p = 0$ corresponds to the standard t-structure on $D_{\text{coh}}^b(\mathcal{O}_X)$. An equivalent way of putting the condition in Theorem 13.2 is that the dual function $\hat{p}(x) = \text{codim}(x) - p(x)$ should again be a supporting function. If that is the case, one has the identities

$$\begin{aligned} \hat{p}D_{\text{coh}}^{\leq k}(\mathcal{O}_X) &= \mathbf{R}\mathcal{H}om\left({}^pD_{\text{coh}}^{\geq -k}(\mathcal{O}_X), \mathcal{O}_X\right) \\ \hat{p}D_{\text{coh}}^{\geq k}(\mathcal{O}_X) &= \mathbf{R}\mathcal{H}om\left({}^pD_{\text{coh}}^{\leq -k}(\mathcal{O}_X), \mathcal{O}_X\right), \end{aligned}$$

which means that the duality functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)$ exchanges the two perverse t-structures defined by p and \hat{p} .

Definition 13.3. The heart of the t-structure defined by p is denoted

$${}^p\text{Coh}(\mathcal{O}_X) = {}^pD_{\text{coh}}^{\leq 0}(\mathcal{O}_X) \cap {}^pD_{\text{coh}}^{\geq 0}(\mathcal{O}_X),$$

and is called the abelian category of *p-perverse coherent sheaves*.

We are interested in a special cases of Kashiwara's result, namely that the set of objects $E \in D_{\text{coh}}^b(\mathcal{O}_X)$ with $\text{codim Supp } \mathcal{H}^i(E) \geq 2i$ for all $i \geq 0$ is part of a t-structure on $D_{\text{coh}}^b(\mathcal{O}_X)$. To formalize this idea, define a function

$$m: X \rightarrow \mathbb{Z}, \quad m(x) = \lfloor \tfrac{1}{2} \text{codim}(x) \rfloor.$$

It is easily verified that both m and the dual function

$$\hat{m}: X \rightarrow \mathbb{Z}, \quad \hat{m}(x) = \lceil \tfrac{1}{2} \operatorname{codim}(x) \rceil$$

are supporting functions. As a consequence of Theorem 13.2, m defines a bounded t-structure on $D_{\text{coh}}^b(\mathcal{O}_X)$; objects of the heart ${}^m\text{Coh}(\mathcal{O}_X)$ will be called *m-perverse coherent sheaves*.

Note. We use this letter because m and \hat{m} are as close as one can get to “middle perversity”. There is of course no actual middle perversity for coherent sheaves, because the equality $p = \hat{p}$ cannot hold unless X is a point.

The next lemma follows easily from [Kas04, Lemma 5.5].

Lemma 13.4. *The perverse t-structures defined by m and \hat{m} satisfy*

$$\begin{aligned} {}^m D_{\text{coh}}^{\leq k}(\mathcal{O}_X) &= \{ E \in D_{\text{coh}}^b(X) \mid \operatorname{codim} \operatorname{Supp} \mathcal{H}^i(E) \geq 2(i - k) \text{ for all } i \in \mathbb{Z} \} \\ {}^{\hat{m}} D_{\text{coh}}^{\leq k}(\mathcal{O}_X) &= \{ E \in D_{\text{coh}}^b(X) \mid \operatorname{codim} \operatorname{Supp} \mathcal{H}^i(E) \geq 2(i - k) - 1 \text{ for all } i \in \mathbb{Z} \}. \end{aligned}$$

By duality, this also describes the subcategories with $\geq k$.

Consequently, an object $E \in D_{\text{coh}}^b(\mathcal{O}_X)$ is an m -perverse coherent sheaf precisely when $\operatorname{codim} \operatorname{Supp} \mathcal{H}^i(E) \geq 2i$ and $\operatorname{codim} \operatorname{Supp} R^i \mathcal{H}om(E, \mathcal{O}_X) \geq 2i - 1$ for every integer $i \geq 0$. This shows one more time that the category of m -perverse coherent sheaves is not preserved by the duality functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)$.

Lemma 13.5. *If $E \in {}^m D_{\text{coh}}^{\geq 0}(\mathcal{O}_X)$, then $E \in D_{\text{coh}}^{\geq 0}(\mathcal{O}_X)$.*

Proof. This is obvious from the fact that $m \geq 0$. □

When it happens that both E and $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)$ are m -perverse coherent sheaves, E has surprisingly good properties.

Proposition 13.6. *If $E \in {}^m D_{\text{coh}}^{\leq 0}(\mathcal{O}_X)$ satisfies $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X) \in {}^m D_{\text{coh}}^{\leq 0}(\mathcal{O}_X)$, then it has the following properties:*

- (i) *Both E and $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)$ belong to ${}^m\text{Coh}(\mathcal{O}_X)$.*
- (ii) *Let $r \geq 0$ be the least integer with $\mathcal{H}^r(E) \neq 0$; then $\operatorname{codim} \operatorname{Supp} \mathcal{H}^r(E) = 2r$.*

Proof. The first assertion follows directly from Lemma 13.4. To prove the second assertion, note that we have $\operatorname{codim} \operatorname{Supp} \mathcal{H}^r(E) \geq 2r$. It therefore suffices to show that if $\mathcal{H}^i(E) = 0$ for $i < r$, and $\operatorname{codim} \operatorname{Supp} \mathcal{H}^r(E) > 2r$, then $\mathcal{H}^r(E) = 0$. Under these assumptions, we have

$$\operatorname{codim} \operatorname{Supp} \mathcal{H}^i(E) \geq \max(2i, 2r + 1) \geq i + r + 1,$$

and therefore $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X) \in D_{\text{coh}}^{\geq r+1}(\mathcal{O}_X)$ by [Kas04, Proposition 4.3]. The same argument, applied to $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)$, now shows that $E \in D_{\text{coh}}^{\geq r+1}(\mathcal{O}_X)$. □

14. Codimension bounds. In this section, we show that the standard t-structure on $D_h^b(\mathcal{D}_A)$ corresponds, under the Fourier-Mukai transform FM_A , to the m -perverse t-structure.

Theorem 14.1. *Let $\mathcal{M} \in D_h^b(\mathcal{D}_A)$ be a holonomic complex on A . Then one has*

$$\begin{aligned} \mathcal{M} \in D_h^{\leq k}(\mathcal{D}_A) &\iff \text{FM}_A(\mathcal{M}) \in {}^m D_{\text{coh}}^{\leq k}(\mathcal{O}_{A^\natural}), \\ \mathcal{M} \in D_h^{\geq k}(\mathcal{D}_A) &\iff \text{FM}_A(\mathcal{M}) \in {}^m D_{\text{coh}}^{\geq k}(\mathcal{O}_{A^\natural}). \end{aligned}$$

The first step of the proof consists in the following “generic vanishing theorem” for holonomic \mathcal{D}_A -modules. In the regular case, this result is due to Krämer and Weissauer [KW11, Theorem 2], whose proof relies on the (difficult) recent solution of Kashiwara’s conjecture for semisimple perverse sheaves. By contrast, our proof is completely elementary.

Proposition 14.2. *Let \mathcal{M} be a holonomic \mathcal{D}_A -module. Then for every $i > 0$, the support of the coherent sheaf $\mathcal{H}^i \text{FM}_A(\mathcal{M})$ is a proper subset of A^\natural .*

Proof. Let $F_\bullet \mathcal{M}$ be a good filtration by \mathcal{O}_A -coherent subsheaves; this exists by [HTT08, Theorem 2.3.1]. As in §9, we consider the associated coherent \mathcal{D}_A -module $R_F \mathcal{M}$ defined by the Rees construction, and its Fourier-Mukai transform

$$\widetilde{\text{FM}}_A(R_F \mathcal{M}) \in \text{D}^b(\mathcal{O}_{E(A)}).$$

By Proposition 9.1, $\widetilde{\text{FM}}_A(R_F \mathcal{M})$ is equivariant for the \mathbb{C}^* -action on $E(A)$, and its restriction to $\lambda^{-1}(1) = A^\natural$ is isomorphic to $\text{FM}_A(\mathcal{M})$. It is therefore sufficient to prove that the restriction of $\widetilde{\text{FM}}_A(R_F \mathcal{M})$ to $\lambda^{-1}(0) = A \times H^0(A, \Omega_A^1)$ has the asserted property. By Proposition 9.1, this restriction is isomorphic to

$$(14.3) \quad \mathbf{R}(p_{23})_* \left(p_{12}^* P \otimes p_{13}^* \mathcal{C}(\mathcal{M}, F) \otimes p_1^* \Omega_A^g \right),$$

where the notation is as in the following diagram:

$$\begin{array}{ccc} A \times H^0(A, \Omega_A^1) & \xleftarrow{p_{13}} & A \times \widehat{A} \times H^0(A, \Omega_A^1) \xrightarrow{p_{23}} \widehat{A} \times H^0(A, \Omega_A^1) \\ & & \downarrow p_{12} \\ & & A \times \widehat{A} \end{array}$$

But \mathcal{M} is holonomic, and so each irreducible component of the support of $\mathcal{C}(\mathcal{M}, F)$ has dimension g . Thus the restriction of p_{23} to the support of $p_{13}^* \mathcal{C}(\mathcal{M}, F)$ is generically finite over $\widehat{A} \times H^0(A, \Omega_A^1)$, which implies that the support of the higher direct image sheaves in (14.3) is a proper subset of $\widehat{A} \times H^0(A, \Omega_A^1)$. \square

Together with the structure theory for cohomology support loci and basic properties of the Fourier-Mukai transform, this result now allows us to prove the first equivalence asserted in Theorem 14.1.

Lemma 14.4. *For any $\mathcal{M} \in \text{D}_h^{\leq k}(\mathcal{D}_A)$, one has $\text{FM}_A(\mathcal{M}) \in {}^m\text{D}_{coh}^{\leq k}(\mathcal{O}_{A^\natural})$.*

Proof. The proof is by induction on $\dim A$, the statement being obviously true when A is a point. Since FM_A is triangulated, it suffices to prove the statement for $k = 0$. According to Lemma 13.4, what we then need to show is the following: for any holonomic complex $\mathcal{M} \in \text{D}_h^{\leq 0}(\mathcal{D}_A)$ concentrated in nonpositive degrees, the Fourier-Mukai transform $\text{FM}_A(\mathcal{M})$ satisfies, for every $\ell \geq 1$, the inequality

$$\text{codim Supp } \mathcal{H}^\ell \text{FM}_A(\mathcal{M}) \geq 2\ell.$$

Let Z be any irreducible component of the support of $\mathcal{H}^\ell \text{FM}_A(\mathcal{M})$, for some $\ell \geq 1$. By (11.4) and descending induction on ℓ , we may assume that Z is also an irreducible component of $S^\ell(\mathcal{M})$; according to Theorem 3.3, Z is therefore a linear subvariety of A^\natural , and hence of the form $Z = t_{(L, \nabla)}(\text{im } f^\natural)$ for a surjective morphism $f: A \rightarrow B$ and a suitable point $(L, \nabla) \in A^\natural$. Furthermore, Proposition 14.2 shows that $\text{codim } Z > 0$, and therefore $\dim B < \dim A$. Setting $r = \dim A - \dim B > 0$, we thus have $\text{codim } Z = 2r$.

Using the properties of the Fourier-Mukai transform listed in Theorem 5.4, we find that the pullback of $\text{FM}_A(\mathcal{M})$ to the subvariety Z is isomorphic to

$$\mathbf{L}(f^\natural)^* \mathbf{L}t_{(L, \nabla)}^* \text{FM}_A(\mathcal{M}) \simeq \text{FM}_B \left(f_+ (\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla)) \right) \in \text{D}_{coh}^b(\mathcal{O}_{B^\natural}).$$

From the definition of the direct image functor f_+ , it is clear that $f_+(\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla))$ belongs to the subcategory $\text{D}_h^{\leq r}(\mathcal{D}_B)$. The inductive assumption now allows us to conclude that the restriction of $\text{FM}_A(\mathcal{M})$ to Z lies in the subcategory ${}^m\text{D}_{coh}^{\leq r}(\mathcal{O}_Z)$.

But Z is an irreducible component of $\text{Supp } \mathcal{H}^\ell \text{FM}_A(\mathcal{M})$; it follows that $\ell \leq r$, and consequently $\text{codim } Z \geq 2\ell$, as asserted. \square

Lemma 14.5. *Let $\mathcal{M} \in \text{D}_h^b(\mathcal{D}_A)$ be a holonomic complex. If its Fourier-Mukai transform satisfies $\text{FM}_A(\mathcal{M}) \in \text{D}_{\text{coh}}^{\leq k}(\mathcal{O}_{A^\natural})$, then $\mathcal{M} \in \text{D}_h^{\leq k}(\mathcal{D}_A)$.*

Proof. It suffices to prove this for $k = 0$. By [Lau96, Théorème 3.2.1], we can recover \mathcal{M} (up to canonical isomorphism) from its Fourier-Mukai transform as

$$\mathcal{M} = \langle -1_A \rangle^* \mathbf{R}(p_1)_* \left(P^\natural \otimes_{\mathcal{O}_{A \times A^\natural}} p_2^* \text{FM}_A(\mathcal{M}) \right) [g],$$

where $p_1: A \times A^\natural \rightarrow A$ and $p_2: A \times A^\natural \rightarrow A^\natural$ are the two projections. If we forget about the \mathcal{D}_A -module structure and only consider \mathcal{M} as a complex of quasi-coherent sheaves of \mathcal{O}_A -modules, we can use the fact that $\pi: A^\natural \rightarrow A$ is affine to obtain

$$\mathcal{M} = \langle -1_A \rangle^* \mathbf{R}(p_1)_* \left(P \otimes_{\mathcal{O}_{A \times \hat{A}}} p_2^* \pi_* \text{FM}_A(\mathcal{M}) \right) [g],$$

where now $p_1: A \times \hat{A} \rightarrow A$ and $p_2: A \times \hat{A} \rightarrow \hat{A}$. By virtue of (11.4) and Theorem 3.3, each irreducible component of $\text{Supp } \mathcal{H}^\ell \text{FM}_A(\mathcal{M})$ is contained in a linear subvariety of codimension at least 2ℓ ; consequently, each irreducible component of $\text{Supp } \pi_* \mathcal{H}^\ell \text{FM}_A(\mathcal{M})$ still has codimension at least ℓ . From this, it is easy to see that $\mathcal{H}^i \mathcal{M} = 0$ for $i > 0$, and hence that $\mathcal{M} \in \text{D}_h^{\leq 0}(\mathcal{D}_A)$. \square

Proof of Theorem 14.1. The first equivalence is proved in Lemma 14.4 and Lemma 14.5 above. The second equivalence follows from this by duality, using the compatibility of the Fourier-Mukai transform with the duality functors for \mathcal{D}_A -modules and \mathcal{O}_{A^\natural} -modules (see Theorem 5.4). \square

15. Proofs for constructible complexes. For the convenience of the reader, we collect in this section the proofs for the results announced in §2. We begin with the structure of the cohomology support loci

$$S_m^k(E) = \left\{ \rho \in \text{Char}(A) \mid \dim \mathbf{H}^k(A, E \otimes_{\mathbb{C}} \mathbb{C}_\rho) \geq m \right\}$$

of a constructible complex $E \in \text{D}_c^b(\mathbb{C}_A)$.

Proof of Theorem 2.2. To prove (a), we use the Riemann-Hilbert correspondence to find a regular holonomic complex $\mathcal{M} \in \text{D}_{rh}^b(\mathcal{D}_A)$ such that $\text{DR}_A(\mathcal{M}) \simeq E$. Since $S_m^k(E) = \Phi(S_m^k(\mathcal{M}))$ by Lemma 11.1, the assertion follows from Theorem 3.3. The statement in (b) may be deduced from Theorem 12.2 by a similar argument. \square

Next comes the description of the perverse t-structure on $\text{D}_c^b(\mathbb{C}_A)$ in terms of the codimension of the loci $S^k(E) = S_1^k(E)$.

Proof of Theorem 2.5. Let $\mathcal{M} \in \text{D}_{rh}^b(\mathcal{D}_A)$ be a regular holonomic complex such that $\text{DR}_A(\mathcal{M}) \simeq E$. Since $\Phi(S^k(\mathcal{M})) = S^k(E)$, the assertion in (a) is a consequence of Theorem 14.1. To deduce (b), let $\mathbf{D}_A: \text{D}_c^b(\mathbb{C}_A) \rightarrow \text{D}_c^b(\mathbb{C}_A)$ be the Verdier duality functor. We then have

$$S_m^k(E) = \langle -1_{\text{Char}(A)} \rangle S_m^{-k}(\mathbf{D}_A E)$$

by Verdier duality. Since $E \in {}^\pi \text{D}_c^{\geq 0}(\mathbb{C}_A)$ iff $\mathbf{D}_A E \in {}^\pi \text{D}_c^{\leq 0}(\mathbb{C}_A)$, the assertion now follows from (a). Finally, (c) is clear from the definition of perverse sheaves as the heart of the perverse t-structure on $\text{D}_c^b(\mathbb{C}_A)$. \square

Lastly, we give the proof of the structure theorem for simple perverse sheaves with Euler characteristic equal to zero.

Proof of Theorem 2.7. This again follows from the Riemann-Hilbert correspondence and the analogous result for simple holonomic \mathcal{D}_A -modules in Corollary 3.11. \square

D. THE STRUCTURE OF FOURIER-MUKAI TRANSFORMS

16. Simple objects. In this section, we prove a structure theorem for the Fourier-Mukai transform of a simple holonomic \mathcal{D}_A -module.

Theorem 16.1. *Let \mathcal{M} be a simple holonomic \mathcal{D}_A -module, and let $r \geq 0$ be the least integer such that $\mathcal{H}^r(\mathrm{FM}_A(\mathcal{M})) \neq 0$. Then there is an abelian variety B of dimension $\dim B = \dim A - r$, a surjective morphism $f: A \rightarrow B$ with connected fibers, and a simple holonomic \mathcal{D}_B -module \mathcal{N} , such that*

$$\mathcal{M} \simeq f^*\mathcal{N} \otimes_{\mathcal{O}_A} (L, \nabla)$$

for a suitable point $(L, \nabla) \in A^\natural$. Moreover, we have $\mathrm{Supp} \mathcal{H}^0(\mathrm{FM}_B(\mathcal{N})) = B^\natural$.

This result clearly implies Theorem 3.10 from the introduction. Here is the proof of the corollary about simple holonomic \mathcal{D}_A -modules with Euler characteristic zero.

Proof of Corollary 3.11. Let $(L, \nabla) \in A^\natural$ be a generic point. Since

$$0 = \chi(A, \mathcal{M}) = \chi(A, \mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla)) = \dim \mathbf{H}^0\left(A, \mathrm{DR}_A(\mathcal{M} \otimes_{\mathcal{O}_A} (L, \nabla))\right),$$

we find that the support of $\mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$ is a proper subset of A^\natural . Since both $\mathrm{FM}_A(\mathcal{M})$ and the dual complex belong to ${}^m\mathrm{D}_{\mathrm{coh}}^{\leq 0}(\mathcal{O}_{A^\natural})$ by Theorem 14.1, we conclude from Proposition 13.6 that $\mathcal{H}^0 \mathrm{FM}_A(\mathcal{M}) = 0$. Now it only remains to apply Theorem 16.1. \square

For the proof of Theorem 16.1, we need two small lemmas. The first describes the inverse image of a simple holonomic \mathcal{D} -module.

Lemma 16.2. *Let $f: A \rightarrow B$ be a surjective morphism of abelian varieties, with connected fibers. If \mathcal{N} is a simple holonomic \mathcal{D}_B -module, then $f^*\mathcal{N}$ is a simple holonomic \mathcal{D}_A -module.*

Proof. Since f is smooth, $f^*\mathcal{N} = \mathcal{O}_A \otimes_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{N}$ is a holonomic \mathcal{D}_A -module. By the classification of simple holonomic \mathcal{D} -modules [HTT08, Theorem 3.4.2], there is a locally closed subvariety $U \subseteq B$, and an irreducible representation $\rho: \pi_1(U) \rightarrow \mathrm{GL}(V)$, such that \mathcal{N} is the minimal extension of the integrable connection on U associated to ρ . Now f has connected fibers, and so the map on fundamental groups

$$f_*: \pi_1(f^{-1}(U)) \rightarrow \pi_1(U)$$

is surjective. Clearly, the pullback $f^*\mathcal{N}$ is equal, over $f^{-1}(U)$, to the integrable connection associated to the representation $\rho \circ f_*: \pi_1(f^{-1}(U)) \rightarrow \pi_1(U) \rightarrow \mathrm{GL}(V)$. This representation is still irreducible because f_* is surjective; to conclude the proof, we shall argue that $f^*\mathcal{N}$ is the minimal extension.

By [HTT08, Theorem 3.4.2], it suffices to show that $f^*\mathcal{N}$ has no submodules or quotient modules that are supported outside of $f^{-1}(U)$. Suppose that $\mathcal{M} \hookrightarrow f^*\mathcal{N}$ is such a submodule. We have $f^+\mathcal{N} = f^*\mathcal{N}[r]$, where $r = \dim A - \dim B$; by adjunction, the morphism $\mathcal{M} \hookrightarrow f^*\mathcal{N}$ corresponds to a morphism $f_+\mathcal{M}[r] \rightarrow \mathcal{N}$, which factors uniquely as

$$f_+\mathcal{M}[r] \rightarrow \mathcal{H}^r f_+\mathcal{M} \rightarrow \mathcal{N}.$$

Since $\mathcal{H}^r f_+\mathcal{M}$ is supported outside of U , this morphism must be zero; consequently, $\mathcal{M} = 0$. A similar result for quotient modules can be derived by applying the duality functor, using [HTT08, Theorem 2.7.1]. This shows that $f^*\mathcal{N}$ is the minimal extension of a simple integrable connection, and hence simple. \square

The second lemma deals with restriction to an irreducible component of the support of a complex.

Lemma 16.3. *Let X be a scheme, and let $E \in D_{coh}^b(\mathcal{O}_X)$. Suppose that Z is an irreducible component of the support of $\mathcal{H}^r(E)$, but not of any $\mathcal{H}^i(E)$ with $i > r$. Let $i: Z \hookrightarrow X$ be the inclusion. Then the morphism*

$$\mathcal{H}^r(E) \rightarrow \mathcal{H}^r(\mathbf{R}i_*\mathbf{L}i^*E)$$

induced by adjunction is nonzero at the generic point of Z .

Proof. After localizing at the generic point of Z , we may assume that $X = \text{Spec } R$ for a local ring (R, \mathfrak{m}) , and that $E \in D_{coh}^b(R)$ is represented by a complex

$$\cdots \rightarrow E^{r-2} \xrightarrow{d} E^{r-1} \xrightarrow{d} E^r$$

of finitely generated free R -modules. Set $M = \mathcal{H}^r(E) = E^r/dE^{r-1}$, which is a finitely generated R -module with $M \neq 0$. Then $\mathcal{H}^r(\mathbf{R}i_*\mathbf{L}i^*E) \simeq M/\mathfrak{m}M$, and the morphism $M \rightarrow M/\mathfrak{m}M$ is nonzero by Nakayama's lemma. \square

We can now prove our structure theorem for simple holonomic \mathcal{D}_A -modules.

Proof of Theorem 16.1. Let $E = \text{FM}_A(\mathcal{M}) \in D_{coh}^b(\mathcal{O}_{A^\natural})$. Theorem 14.1 shows that $E \in {}^m\text{Coh}(\mathcal{O}_{A^\natural})$; by duality, it follows that $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_{A^\natural}) \in {}^m\text{Coh}(\mathcal{O}_{A^\natural})$, too. According to Proposition 13.6, the codimension of the support of $\mathcal{H}^r(E)$ is therefore equal to $2r$; moreover, each irreducible component of $\text{Supp } \mathcal{H}^r(E)$ of codimension $2r$ is also an irreducible component of $S^r(\mathcal{M})$ by (11.4). After tensoring \mathcal{M} by a suitable line bundle with integrable connection, we may assume that one irreducible component of the support of $\mathcal{H}^r(E)$ is equal to $\text{im } f^\natural$, for a surjective morphism of abelian varieties $f: A \rightarrow B$ with connected fibers and $\dim B = \dim A - r$.

To produce the required simple \mathcal{D}_B -module, consider the direct image $f_+\mathcal{M}$, which belongs to $D_h^{\leq r}(\mathcal{D}_B)$. We have a distinguished triangle

$$\tau_{\leq r-1}(f_+\mathcal{M}) \rightarrow f_+\mathcal{M} \rightarrow \mathcal{H}^r(f_+\mathcal{M})[-r] \rightarrow \cdots$$

in $D_h^b(\mathcal{D}_B)$, and hence also a distinguished triangle

$$(16.4) \quad f^+\tau_{\leq r-1}(f_+\mathcal{M}) \rightarrow f^+f_+\mathcal{M} \rightarrow f^+\mathcal{H}^r(f_+\mathcal{M})[-r] \rightarrow \cdots$$

in $D_h^b(\mathcal{D}_A)$. Since f is smooth, $f^+\mathcal{H}^r(f_+\mathcal{M})[-r] = f^*\mathcal{H}^r(f_+\mathcal{M})$ is a single holonomic \mathcal{D}_A -module. Let $\alpha: \mathcal{M} \rightarrow f^+\mathcal{H}^r(f_+\mathcal{M})$ be the adjunction morphism.

Now we observe that the induced morphism $\mathcal{M} \rightarrow f^*\mathcal{H}^r(f_+\mathcal{M})$ must be nonzero. Indeed, suppose to the contrary that the morphism was zero. Then α factors as

$$\mathcal{M} \rightarrow f^+\tau_{\leq r-1}(f_+\mathcal{M}) \rightarrow f^+f_+\mathcal{M}.$$

If we apply the Fourier-Mukai transform to this factorization, and use the properties in Theorem 5.4, we obtain

$$E \rightarrow \mathbf{R}f_*^\natural \text{FM}_B(\tau_{\leq r-1}(f_+\mathcal{M})) \rightarrow \mathbf{R}f_*^\natural \mathbf{L}(f^\natural)^*E,$$

which is a factorization of the adjunction morphism for the closed embedding f^\natural . In particular, we then have

$$\mathcal{H}^r(E) \rightarrow \mathcal{H}^r(\mathbf{R}f_*^\natural \text{FM}_B(\tau_{\leq r-1}(f_+\mathcal{M}))) \rightarrow \mathcal{H}^r(\mathbf{R}f_*^\natural \mathbf{L}(f^\natural)^*E);$$

but because the coherent sheaf in the middle is supported in a subset of $\text{im } f^\natural$ of codimension at least two, this contradicts Lemma 16.3. Therefore, $\mathcal{M} \rightarrow f^*\mathcal{H}^r(f_+\mathcal{M})$ is indeed nonzero.

Being a holonomic \mathcal{D}_B -module, $\mathcal{H}^r(f_+\mathcal{M})$ admits a finite filtration with simple quotients; consequently, we can find a simple holonomic \mathcal{D}_B -module \mathcal{N} and a nonzero morphism $\mathcal{M} \rightarrow f^*\mathcal{N}$. Since \mathcal{M} is simple, and $f^*\mathcal{N}$ is also simple by Lemma 16.2, the morphism must be an isomorphism, and so $\mathcal{M} \simeq f^*\mathcal{N}$.

To prove the final assertion, note that $f^*\mathcal{N} = f^+\mathcal{N}[-r]$; on account of Theorem 5.4, we therefore have

$$\mathrm{FM}_A(\mathcal{M}) \simeq \mathrm{FM}_A(f^*\mathcal{N}) \simeq \mathbf{R}f_*^{\natural} \mathrm{FM}_B(\mathcal{N})[-r].$$

Since $\mathrm{im} f^{\natural}$ is an irreducible component of the support of $\mathcal{H}^r(\mathrm{FM}_A(\mathcal{M}))$, it follows that $\mathrm{Supp} \mathcal{H}^0(\mathrm{FM}_B(\mathcal{N})) = B^{\natural}$, as claimed. \square

17. Chern characters. The purpose of this section is to compute the algebraic Chern character of $\mathrm{FM}_A(\mathcal{M})$, for \mathcal{M} a holonomic \mathcal{D}_A -module.

For a smooth algebraic variety X , we denote by $CH(X)$ the algebraic Chow ring of X . To begin with, observe that since $\pi: A^{\natural} \rightarrow \hat{A}$ is an affine bundle in the Zariski topology, the pullback map $\pi^*: CH(\hat{A}) \rightarrow CH(A^{\natural})$ is an isomorphism.

Proposition 17.1. *Let \mathcal{M} be a holonomic \mathcal{D}_A -module. Then the algebraic Chern character of the Fourier-Mukai transform $\mathrm{FM}_A(\mathcal{M})$ lies in the subring of $CH(\hat{A})$ generated by $CH_1^1(\hat{A}) = \mathrm{Pic}^0(\hat{A})$.*

Proof. Since $\pi: E(A) \rightarrow \hat{A}$ is an algebraic vector bundle containing $A^{\natural} = \lambda^{-1}(1)$, pullback of cycles induces isomorphisms

$$CH(\hat{A}) \simeq CH(E(A)) \simeq CH(A^{\natural}).$$

As in the proof of Proposition 14.2, choose a good filtration $F_{\bullet}\mathcal{M}$ and consider the Fourier-Mukai transform $\widetilde{\mathrm{FM}}_A(R_F\mathcal{M})$ of the associated Rees module. Its restriction to A^{\natural} is isomorphic to $\mathrm{FM}_A(\mathcal{M})$, and so it suffices to show that the Chern character of $\widetilde{\mathrm{FM}}_A(R_F\mathcal{M})$ is contained in the subring generated by $\mathrm{Pic}^0(\hat{A})$. Since $\lambda^{-1}(0) = \hat{A} \times H^0(A, \Omega_A^1)$, we only need to prove this after restricting to $\hat{A} \times \{\omega\} \subseteq \lambda^{-1}(0)$, for any choice of $\omega \in H^0(A, \Omega_A^1)$.

By Proposition 9.1, the restriction of $\widetilde{\mathrm{FM}}_A(R_F\mathcal{M})$ to $\lambda^{-1}(0)$ is isomorphic to

$$(17.2) \quad \mathbf{R}(p_{23})_* \left(p_{12}^* P \otimes p_1^* \Omega_A^g \otimes p_{13}^* \mathcal{C}(\mathcal{M}, F) \right).$$

Since \mathcal{M} is holonomic, the support of $\mathcal{C}(\mathcal{M}, F)$ is of pure dimension g . Now choose $\omega \in H^0(A, \Omega_A^1)$ general enough that the restriction of $\mathcal{C}(\mathcal{M}, F)$ to $A \times \{\omega\}$ is a coherent sheaf with zero-dimensional support. The restriction of (17.2) to $\hat{A} \times \{\omega\}$ is then the Fourier-Mukai transform of a coherent sheaf on A with zero-dimensional support; its algebraic Chern character must therefore be contained in the subring of $CH(\hat{A})$ generated by $A = \mathrm{Pic}^0(\hat{A})$. \square

Corollary 17.3. *Let \mathcal{M} be a holonomic \mathcal{D}_A -module. Then all the Chern classes of $\mathrm{FM}_A(\mathcal{M})$ are zero in the singular cohomology ring of A^{\natural} .*

REFERENCES

- [Ara92] D. Arapura, *Higgs line bundles, Green-Lazarsfeld sets, and maps of Kähler manifolds to curves*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), no. 2, 310–314.
- [AB10] D. Arinkin and R. Bezrukavnikov, *Perverse coherent sheaves*, Mosc. Math. J. **10** (2010), no. 1, 3–29.
- [BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171 (French).
- [Bon10] J. Bönzsdorff, *Autodual connection in the Fourier transform of a Higgs bundle*, Asian J. Math. **14** (2010), no. 2, 153–173.
- [Dim04] A. Dimca, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004.
- [FK00] J. Franécki and M. Kapranov, *The Gauss map and a noncompact Riemann-Roch formula for constructible sheaves on semiabelian varieties*, Duke Math. J. **104** (2000), no. 1, 171–180.

- [GL87] M. Green and R. Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math. **90** (1987), no. 2, 389–407.
- [GL91] ———, *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. **1** (1991), no. 4, 87–103.
- [HTT08] R. Hotta, K. Takeuchi, and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008.
- [Kas04] M. Kashiwara, *t-structures on the derived categories of holonomic \mathcal{D} -modules and coherent \mathcal{O} -modules*, Moscow Math. J. **4** (2004), no. 4, 847–868.
- [KW11] T. Krämer and R. Weissauer, *Vanishing theorems for constructible sheaves on abelian varieties* (2011), available at [arXiv:1111.4947v3](https://arxiv.org/abs/1111.4947v3).
- [Lau96] G. Laumon, *Transformation de Fourier généralisée* (1996), available at [arXiv:alg-geom/9603004](https://arxiv.org/abs/alg-geom/9603004).
- [MM74] B. Mazur and W. Messing, *Universal Extensions and One Dimensional Crystalline Cohomology*, Lecture Notes in Math., vol. 370, Springer-Verlag, Berlin, 1974.
- [Moc10] T. Mochizuki, *Holonomic D-module with Betti structure* (2010), available at [arXiv:1001.2336v3](https://arxiv.org/abs/1001.2336v3).
- [PR01] A. Polishchuk and M. Rothstein, *Fourier transform for D-algebras. I*, Duke Math. J. **109** (2001), no. 1, 123–146.
- [PS11] M. Popa and C. Schnell, *Generic vanishing theory via mixed Hodge modules* (2011), available at [arXiv:1112.3058](https://arxiv.org/abs/1112.3058).
- [Rot96] M. Rothstein, *Sheaves with connection on abelian varieties*, Duke Math. J. **84** (1996), no. 3, 565–598.
- [Sai91] M. Saito, *Hodge conjecture and mixed motives. I*, Complex geometry and Lie theory (Sundance, UT, 1989), Proc. Sympos. Pure Math., vol. 53, Amer. Math. Soc., Providence, RI, 1991, pp. 283–303.
- [Sim93] C. Simpson, *Subspaces of moduli spaces of rank one local systems*, Ann. Sci. ENS **26** (1993), 361–401.
- [Sim94] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 47–129.
- [Wat60] C. E. Watts, *Intrinsic characterizations of some additive functors*, Proc. Amer. Math. Soc. **11** (1960), 5–8.
- [Wei12] R. Weissauer, *Degenerate Perverse Sheaves on Abelian Varieties* (2012), available at [arXiv:1204.2247](https://arxiv.org/abs/1204.2247).

INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE, THE UNIVERSITY OF TOKYO,
 5-1-5 KASHIWANOHA, KASHIWA-SHI, CHIBA 277-8583, JAPAN
E-mail address: christian.schnell@ipmu.jp